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RESEARCH ON THE SONIC BOOM PROBLEM

Part 1 - Second-Order Solutions for the Flow Field Around Slender Bodies in Supersonic Flow for Sonic Boom Analysis

by M. Landahl and P. Lofgren

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THE AERONAUTICAL RESEARCH INSTITUTE OF SWEDEN
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16. Abstract

A second-order theory for supersonic flow past slender bodies is presented. Through the introduction of characteristic coordinates as independent variables and the expansion procedure proposed by Lin and Oswatitsch, a uniformly valid solution is obtained for the whole flow field in the axisymmetric case and for the far field in the general three-dimensional case. For distances far from the body the theory is an extension of Whitham's first-order solution and for the domain close to the body it is a modification of Van Dyke's second-order solution in the axisymmetric case. From the theory useful formulas relating flow deflections to the Whitham F-function are derived, which permits one to determine the sonic boom strength from wind tunnel measurements fairly close to the body.

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NOMENCLATURE

English alphabet

A, A', A_o and A'_o	see Fig. 3.2 and p. 18
A_n, $n = 1,2,3.$	see p. 40
a_{ij}, $i,j = 1,2,3.$	see p. 39
B	$=\sqrt{M^2 - 1}$
B_L	$=\sqrt{M_L^2 - 1}$
B_n, $n = 1,2,3.$	see (B.7)
b_{ij}, $i,j = 1,2,3.$	see (B.8)
c	local speed of sound
c_∞	speed of sound in the free stream
$\vec{e}_x$, $\vec{e}_r$, $\vec{e}_\theta$	the basis vectors in a cylindrical coordinate system
$F(\xi)$ $F(\xi, \theta_o)$	Whitham's F-function
$F(\xi, \eta)$	generalized F-function (see (4.10))
$F^*(\xi)$ $F_n^*(\xi)$, $n = 0,1,2,..$	see (C.6)
$\mathscr{F}_n$, $n = 1,2.$	see (D.7), (D.15), (5.6) and (5.21)
$f(t)$	see (C.1) - (C.4)
f_n, $n = 1,2,...,7.$	see (2.9) - (2.15) and (B.23) - (B.29)
G_L	see p. 7
H	$= 2Br\varepsilon^4$ except in Appendix B, where H is a column vector (see p. 40)
H_n, $n = 0,1,2,...$	see p. 19 and (5.1)
h	see p. 39
h_n, $n = 0,1.$	see (5.17), (D.5) and (D.13)
K	$= \frac{\gamma+1}{2} \frac{M^4}{B^2}$
k	see p. 41
L	$= \frac{K}{2} (K + \frac{3}{2} M^2 - \frac{2K}{M^2})$

ℓ_n, $n = 0.1$	see (5.18), (D.8) and (D.16)
M	free-stream Mach number
M_L	local Mach number
$\vec{n}$	unit vector normal to the shock
ord	see p. 10
p_n, $n = 1,2,3.$	arbitrary real numbers
$\bar{p}$	row vector (p_1, p_2, p_3)
Q	see p. 40
$\vec{q}$	flow velocity vector
q_1	$= u$
q_2	$= v$
q_3	$= rW$
$\hat{q}$	Bernoulli constant (see p. 5)
$R(x)$, $\bar{R}(x)$	$r = R(x) = \epsilon\bar{R}(x)$ is the body contour in the axisymmetric case
$R_o(x_o)$	$r_o = R_o(x_o)$ is the image of the body contour in the $x_o r_o$-plane
r	cylindrical coordinate
r_n, $n = 0,1,2,\ldots$	see (3.3), (5.29) and (5.34) - (5.35)
r_{oB}	see subscript B below
r_m	see Fig. 6.2
U	$\vec{q} = U\vec{e}_x + V\vec{e}_r + W\vec{e}_\theta$
U_∞	free-stream speed
U_n, $n = 1,2,\ldots$	see p. 19 and (5.1)
u	$U = U_\infty(1 + u)$
$\bar{u}$	row vector (u,v,w)
u_o	see (3.21) and (5.29)
u_n, $n = 1,2,3,\ldots$	see (3.3)
u_B	$= u_{oB}$ see subscript B below

V	see U
V_n, n = 1,2,3,...	see p. 19 and (5.1)
v	$V = U_\infty v$
v_o	see (3.22) and (5.29)
v_n, n = 1,2,3,...	see (3.3)
v_B	$= v_{oB}$ see subscript B below
W	see U
W_n, n = 1,2,3,...	see (5.1)
w	$W = U_\infty w$
w_o	see (5.29)
x	cylindrical coordinate
$\bar{x}$	row vector (x,r,θ)
x_n, n = 0,1,2,3,...	see (3.3), (5.29) and (5.34) - (5.35); in Appendix B $x_1 = x$, $x_2 = r$ and $x_3 = \theta$
x_{oB}	see subscript B below
$x_s(r)$	$x = x_s(r)$ is a shock surface
y_n, n = 1,2,3.	see p. 41
$\bar{y}$	row vector (y_1, y_2, y_3)

Greek alphabet

α	$\sin \alpha = \frac{1}{M_L}$
γ	$= c_p/c_v$
ϵ	overall thickness ratio of the body
ζ	characteristic variable (see p. 44 with $y_1 = \xi$, $y_2 = \eta$ and $y_3 = \zeta$)
η	characteristic variable (see ζ and p. 38)
θ	cylindrical coordinate
θ_n, n = 0,1,2,3,...	see (5.1)

ϑ	flow deflection angle ($V = U \text{ tg } \vartheta$)
ϑ_1	see (5.19)
$\varkappa_\xi$	see (C.7)
$\lambda(\xi)$	see (3.51)
λ_n, $n = 1,2,\ldots,6.$	see (3.8), (3.9), (3.15), (3.16), (3.36), (3.37), (3.40) and (3.53) - (3.55)
$\lambda^{(n)}$, $n = 1,2,3.$	see p. 45
μ	$\mu^2 = \frac{\gamma-1}{\gamma+1}$
ξ	characteristic variable (see ζ and p. 38)
σ, τ	see Fig. 6.1
Φ	velocity potential ($\vec{q} = \nabla\Phi$)
ϕ	$u\vec{e}_x + v\vec{e}_r + w\vec{e}_\theta = \nabla\phi$
ϕ_o	see (3.25) and (5.29)
ϕ_n, $n = 1,2,3,\ldots$	see (3.12) and (3.31)
ϕ_B	$= \phi_{oB}$ see subscript B below
χ	$= x - Br$
χ_n, $n = 0,1,2,\ldots$	see p. 19 and (5.1)
$\psi(r)$	$x = \psi(r)$ is a shock surface
$\psi(\bar{x})$	$\psi(\bar{x})$ = const. is a characteristic surface see (B.9)
$\Omega_\pm$	see (B.11)
$\omega_o(x_o)$	see (C.3)

Additional Symbols

Superscript	$^{(1)}$	denotes quantity in front of the shock
-"-	(2)	-"- -"- behind -"-
-"-	(a)	-"- the corresponding axisymmetric expression
Subscript B		values taken at the body contour (see p. 15)
$\frac{\partial}{\partial \bar{x}}$		row vector $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}\right)$
$\frac{\partial}{\partial \bar{y}}$		-"- $\left(\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \frac{\partial}{\partial y_3}\right)$
$\frac{\partial}{\partial \bar{p}}$		-"- $\left(\frac{\partial}{\partial p_1}, \frac{\partial}{\partial p_2}, \frac{\partial}{\partial p_3}\right)$

$\frac{\partial(\bar{x})}{\partial(\bar{y})}$, $\frac{\partial(y_1,y_2,y_3)}{\partial(x_1,x_2,x_3)}$ and $\frac{\partial(x,r,\theta)}{\partial(\xi,\eta,\zeta)}$ are functional determinants

1. INTRODUCTION

In the well known Whitham theory [8] for predicting the strength of the sonic boom at large distances from a supersonic vehicle only such nonlinear effects are included which produce a distortion of the signal as it travels away from the vehicle. These have a cumulative effect at large distances and are therefore important even for very small disturbance levels. Whitham's simple first-order rule, namely that the perturbation velocity field is the same as the one obtained by linear theory but displaced in the streamwise direction so as to take into account the distortion of the downstream Mach cones, will usually be satisfactory for the very slender configurations of interest for commercial SST's when flying at moderate supersonic Mach numbers. For not so slender configurations or high Mach numbers (above 3, say) higher-order nonlinear effects can be expected to become important in regions close to the vehicle. For larger distances, say a few body lengths away, linear acoustics would still be expected to hold, although the signal strength would not be correctly given by the first-order theory relating signal strength to vehicle geometry.

To assess the importance of such higher-order nonlinear effects in the near field, a second-order theory for the complete flow field would be of great value as it would allow one to estimate the errors inherent in the first-order theory. Such a theory for the near field of a body of revolution at zero angle of attack was obtained some time ago by Van Dyke [7] by the use of a regular perturbation method. This solution is not uniformly valid in the far field domain — or in regions influenced by discontinuities in the body surface slope — since the second-order solution thus obtained has the same set of characteristics (free-stream characteristics), and hence influence regions, as the first-order (linearized) solution. Therefore, Van Dyke's theory needs to be modified so as to account for the change in the characteristics. A systematic expansion procedure yielding a uniformly valid solution in the whole flow field has been proposed by Lin [2] and further extended by Oswatitsch [3]. In this,

both families of characteristics are readjusted at each step. In an earlier publication [13], it was shown that this procedure could be used to cast Van Dyke's second-order solution for axisymmetric flow in a uniformly valid form for moderate and large radii. The present report shows how the uniformly valid solution can be obtained directly by aid of the Lin-Oswatitsch formal expansion procedure. Van Dyke's results are obtained when the solution is expanded in a Taylor series around the free stream characteristics. Thus, an extension of Whitham's [8] rule to second order is presented.

The new theory is also modified to allow for discontinuities in body surface slope. For the determination of shock wave location a simple extension of Whitham's "rule of equal areas" is presented.

The three-dimensional far field is treated in a similar manner using a perturbed cylindrical coordinate system. For the three-dimensional mid and near fields, the use of characteristic variables does not appear to lead to a simple second-order solution. The far-field solution is used to generate formulas relating flow deflections to the Whitham F-function. With the aid of such formulas a new experimental technique was devised [14] for the determination of the sonic boom strength from wind tunnel measurements which, particularly for high Mach numbers, can be carried out fairly close to the body where the disturbance levels are large enough to give good experimental accuracy.

2. GENERAL ASSUMPTIONS, BASIC DIFFERENTIAL EQUATIONS AND CHARACTERISTIC COORDINATES

We study the steady supersonic flow field around a fixed rigid body, placed in a homogeneous parallel stream of a polytropic ideal gas. Viscous forces, gravity and heat conduction are neglected. Cylindrical coordinates x,r and θ are introduced such that the free stream is in the positive x-direction and the body is situated in the half-space $x \geq 0$. Further, it is assumed that all streamlines start from $x = -\infty$.

The entropy change across a weak shock wave is of the order of the shock strength to the third power, i.e. of the order of the cube of the velocity perturbation jump across the wave. In the second-order theory considered here only contributions from terms which are quadratic in the velocity perturbations are retained, except near a slender body of revolution, where some terms which are formally cubic in the v-perturbation are included. For a smooth pointed body, however, the bow shock strength will be of the order u^2 also near the body, so that entropy changes may still be neglected in this region. If the body has slope discontinuities of order ϵ, where ϵ is the overall thickness ratio of the body, the perturbation velocities and hence the shock strength will be of order ϵ, so that entropy changes and the entropy gradients become of order ϵ^3, which may be neglected in the second-order theory then retaining only terms of order ϵ^2. Thus, to the approximation considered, we may neglect the entropy changes and hence consider the flow to be isentropic and irrotational throughout the flow field. We may thus start from the equations for irrotational flow

$$\left\{\begin{aligned}
&(U^2 - c^2)\frac{\partial U}{\partial x} + (V^2 - c^2)\frac{\partial V}{\partial r} + (W^2 - c^2)\frac{\partial W}{r\partial\theta} + UV\left(\frac{\partial V}{\partial x} + \frac{\partial U}{\partial r}\right) + \\
&\quad + UW\left(\frac{\partial W}{\partial x} + \frac{\partial U}{r\partial\theta}\right) + VW\left(\frac{\partial W}{\partial r} + \frac{\partial V}{r\partial\theta}\right) = \frac{c^2 V}{r} && (2.1) \\
&\frac{\partial V}{\partial x} = \frac{\partial U}{\partial r} && (2.2) \\
&\frac{\partial W}{\partial x} = \frac{\partial U}{r\partial\theta} && (2.3) \\
&\frac{\partial W}{\partial r} + \frac{W}{r} = \frac{\partial V}{r\partial\theta} && (2.4)
\end{aligned}\right.$$

with the velocity of sound c given by

$$\frac{1}{2}(U^2 + V^2 + W^2) + \frac{c^2}{\gamma - 1} = \frac{1}{2}\hat{q}^2$$

(Bernoulli's equation).

Here $\hat{q}$ is a constant, $\gamma = C_p/C_v$ and U, V and W are defined by

$$\vec{q} = U\,\vec{e}_x + V\,\vec{e}_r + W\,\vec{e}_\theta$$

$\vec{q}$ being the flow-velocity vector.

The above equations together with the shock relations and the conditions for the flow to be tangent to the body surface and to tend to the free-stream flow for $x \to -\infty$ specify the flow in the whole field.

In the following we consider x and r as dimensionless coordinates, obtained through division by a suitable reference length such as the overall streamwise dimension of the body. This does not change the form of (2.1) - (2.4). We shall also use dimensionless disturbance velocities u, v and w defined by

$$\left.\begin{aligned} U &= U_\infty(1 + u) \\ V &= U_\infty\, v \\ W &= U_\infty\, w \end{aligned}\right\}$$

U_∞ denoting the free-stream speed.

In the axisymmetric case (with the x-axis as symmetry axis) we have $\partial/\partial\theta \equiv W \equiv 0$ so that (2.3) and (2.4) need not be considered. In that case it is advantageous ([2],[3]) to introduce characteristic coordinates ξ and η as new independent variables (see Appendix A). If we choose ξ and η such that ξ = const. is a downstream characteristic and η = const. an upstream characteristic the transformation $(\xi,\eta) \to (x,r)$ can be written

$$\left\{\begin{aligned} &(1 + u - B_L v)\frac{\partial x}{\partial \xi} + (B_L + B_L u + v)\frac{\partial r}{\partial \xi} = 0 \qquad (2.5) \\ \\ &(1 + u + B_L v)\frac{\partial x}{\partial \eta} - (B_L + B_L u - v)\frac{\partial r}{\partial \eta} = 0 \qquad (2.6) \end{aligned}\right.$$

and equations (2.1) and (2.2) are then transformed into

$$\begin{cases} (B_L + B_L u - v)\dfrac{\partial u}{\partial \xi} + (1 + u + B_L v)\dfrac{\partial v}{\partial \xi} + (1 + u + B_L v)(1 + G_L v^2)\dfrac{v}{r}\dfrac{\partial r}{\partial \xi} = 0 & (2.7) \\ (B_L + B_L u + v)\dfrac{\partial u}{\partial \eta} - (1 + u - B_L v)\dfrac{\partial v}{\partial \eta} - (1 + u - B_L v)(1 + G_L v^2)\dfrac{v}{r}\dfrac{\partial r}{\partial \eta} = 0 & (2.8) \end{cases}$$

Here $B_L = \sqrt{M_L^2 - 1}$ and $G_L = M^2\left(1 - \frac{\gamma-1}{2} M^2(2u + u^2) - \frac{\gamma+1}{2} M^2 v^2\right)^{-1}$.

Index L denotes local values and M the free stream Mach number.

From Bernoulli's equation we obtain

$$B_L = B\left[1 + (K - M^2)(2u + u^2 + v^2)\left(1 - \frac{\gamma-1}{2} M^2[2u + u^2 + v^2]\right)^{-1}\right]^{\frac{1}{2}}$$

with $K = \dfrac{\gamma+1}{2}\,\dfrac{M^4}{B^2}$

and $B = \sqrt{M^2 - 1}$.

Each of the equations (2.5) - (2.8) contains derivatives with respect to only one variable. This simplifies the integration of the differential equations considerably.

In the general three-dimensional case one speaks of characteristic surfaces and bicharacteristics belonging to the system (2.1) - (2.4) (see Appendix B). For the special case of undisturbed flow, i.e. $\vec{q} \equiv U_\infty \vec{e}_x$ — which is a trivial solution of (2.1) - (2.4) — the downstream Mach cone $x - Br = \text{const.}$ and the upstream Mach cone $x + Br = \text{const.}$ are characteristic surfaces and the meridian plane $\theta = \text{const.}$ intersects the downstream Mach cone along a bicharacteristic (see Appendix B) In our study of the general three-dimensional case — with the only restriction that the disturbance caused by the body is small — it appears to be advantageous to introduce new independent variables ξ, η and ζ such that

ξ = const. is a characteristic surface and a disturbed downstream Mach cone $(x - Br \approx \text{const.})$,

η = const. is a characteristic surface and a disturbed upstream Mach cone $(x + Br \approx \text{const.})$,

ζ = const. intersects ξ = const. along a bicharacteristic and is a disturbed meridian plane $(\theta \approx \text{const.})$.

In the following, the variables ξ, η and ζ will be referred to as characteristic coordinates. The transformation $(\xi,\eta,\zeta) \rightarrow (x,r,\theta)$ can be written formally (see Appendix B)

$$\begin{cases} f_1(r,\bar{u}, \frac{\partial \bar{x}}{\partial \xi}, \frac{\partial \bar{x}}{\partial \zeta} ; M, \gamma) = 0 & (2.9) \\ \\ f_2(r,\bar{u}, \frac{\partial \bar{x}}{\partial \eta}, \frac{\partial \bar{x}}{\partial \zeta} ; M, \gamma) = 0 & (2.10) \\ \\ f_3(r,\bar{u}, \frac{\partial \bar{x}}{\partial \eta}, \frac{\partial \bar{x}}{\partial \zeta} ; M, \gamma) = 0 & (2.11) \end{cases}$$

Also the equations (2.1) - (2.4) can be transformed to equations of the form

$$\begin{cases} f_4(r,\bar{u}, \frac{\partial \bar{x}}{\partial \xi}, \frac{\partial \bar{x}}{\partial \eta}, \frac{\partial \bar{x}}{\partial \zeta}, \frac{\partial \bar{u}}{\partial \xi}, \frac{\partial \bar{u}}{\partial \zeta} ; M, \gamma) = 0 & (2.12) \\ \\ f_5(r,\bar{u}, \frac{\partial \bar{x}}{\partial \xi}, \frac{\partial \bar{x}}{\partial \eta}, \frac{\partial \bar{x}}{\partial \zeta}, \frac{\partial \bar{u}}{\partial \eta}, \frac{\partial \bar{u}}{\partial \zeta} ; M, \gamma) = 0 & (2.13) \\ \\ f_6(r , \frac{\partial \bar{x}}{\partial \xi}, \frac{\partial \bar{x}}{\partial \eta}, \frac{\partial \bar{x}}{\partial \zeta}, \frac{\partial \bar{u}}{\partial \xi}, \frac{\partial \bar{u}}{\partial \eta}, \frac{\partial \bar{u}}{\partial \zeta}) = 0 & (2.14) \\ \\ f_7(r,w, \frac{\partial \bar{x}}{\partial \xi}, \frac{\partial \bar{x}}{\partial \eta}, \frac{\partial \bar{x}}{\partial \zeta}, \frac{\partial \bar{u}}{\partial \xi}, \frac{\partial \bar{u}}{\partial \eta}, \frac{\partial \bar{u}}{\partial \zeta}) = 0 & (2.15) \end{cases}$$

Here $\bar{x}$ and $\bar{u}$ denote the vector quantities (x,r,θ) and (u,v,w), respectively, and the f_n's are complicated expressions given by the left-hand sides of equations (B.23) - (B.29). The following can be said about (2.9) - (2.15):

Equations (2.9) and (2.10) are the conditions for η = const. and ξ = const., respectively, to be characteristic surfaces. In case of axisymmetry they are equivalent to (2.5) and (2.6), respectively.

Equation (2.11) is the condition for ζ = const. to intersect ξ = const. along a bicharacteristic. In the case of axisymmetry equations (2.12) and (2.13) are equivalent to (2.7) and (2.8), respectively. Similarly (2.14) and (2.15) are just (2.3) and (2.4), respectively, (f_6 does not contain $\partial v/\partial\xi$, $\partial v/\partial\eta$, and $\partial v/\partial\zeta$ and f_7 does not contain $\partial u/\partial\xi$, $\partial u/\partial\eta$, and $\partial u/\partial\zeta$).

As in the axisymmetric case, absence of certain derivatives in (2.9) - (2.15) helps simplify the integration procedure.

3. A UNIFORMLY VALID SECOND-ORDER SOLUTION FOR AXISYMMETRIC FLOW AROUND A SLENDER BODY OF REVOLUTION

Let the body considered be located in the half-space $x \geq 0$ with the positive x-axis as axis of symmetry and let its contour be described by

$$r = R(x) = \epsilon\bar{R}(x) \qquad (3.1)$$

where $\bar{R} = O(1)$. The body is assumed to be pointed, and is allowed to have slope discontinuities of order ϵ. The flow must satisfy the tangency condition at the body surface,

$$v = R'(1 + u) \qquad (3.2)$$

and must be undisturbed upstream of the body. In addition, shock waves give rise to boundary conditions on surfaces $x = x_s(r)$ which

cannot be specified a priori. It turns out that shocks lead to a nonuniqueness in the transformation $(\xi,\eta)\rightarrow(x,r)$ which requires special treatment as shown in the subsequent section.

In attacking the problem for axisymmetric flow we shall first determine a second-order solution valid in the mid-field region r ord (1) *) and then extend this to regions near the body and far from the body.

For the mid-field expansion we assume, following the procedure proposed by Lin [2] and further extended by Oswatitsch [3],

$$\begin{aligned} u &= u_1 + u_2 + \dots \\ v &= v_1 + v_2 + \dots \\ x &= x_o + x_1 + \dots \\ r &= r_o + r_1 + \dots \end{aligned} \tag{3.3}$$

where the different terms are to be regarded as functions of the characteristic coordinates ξ,η. Here, x_o and r_o are of order unity, u_1, v_1, x_1 and r_1 of order ϵ^2. Upon substitution of (3.3) into (2.5)-(2.8) we obtain

$$\frac{\partial}{\partial\xi}(x_o + B\,r_o) = 0 \tag{3.4}$$

$$\frac{\partial}{\partial\eta}(x_o - B\,r_o) = 0 \tag{3.5}$$

$$\frac{\partial}{\partial\xi}(B\,u_1 + v_1) + \frac{v_1}{r_o}\frac{\partial r_o}{\partial\xi} = 0 \tag{3.6}$$

$$\frac{\partial}{\partial\eta}(B\,u_1 - v_1) - \frac{v_1}{r_o}\frac{\partial r_o}{\partial\eta} = 0 \tag{3.7}$$

---

*) By r = ord (1) we mean $\begin{cases} r = O(1) \\ r^{-1} = O(1) \end{cases}$

The solutions of (3.4) and (3.5) read

$$x_o + B\,r_o = \lambda_1(\eta) \tag{3.8}$$

$$x_o - B\,r_o = \lambda_2(\xi) \tag{3.9}$$

Without loss of generality we may choose $\lambda_1 = \eta$ and $\lambda_2 = \xi$. The characteristics will appear as straight lines in the $x_o r_o$-plane, and it may be convenient, following Oswatitsch [3], to use

$$x_o = \frac{1}{2}(\eta + \xi) \tag{3.10}$$

$$r_o = \frac{1}{2\,B}(\eta - \xi) \tag{3.11}$$

in lieu of ξ and η as characteristic coordinates.

Substituting (3.10) and (3.11) into (3.6) and (3.7) we obtain the usual linearized equation for u_1 and v_1, i.e.

$$-B^2\frac{\partial^2\phi_1}{\partial x_o^2} + \frac{1}{r_o}\frac{\partial\phi_1}{\partial r_o} + \frac{\partial^2\phi_1}{\partial r_o^2} = 0 \tag{3.12}$$

where $u_1 = \partial\phi_1/\partial x_o$, $v_1 = \partial\phi_1/\partial r_o$.

We proceed now to next order. From (2.5) and (2.6) we obtain

$$\frac{\partial}{\partial\xi}(x_1 + B\,r_1) + (u_1 - B\,v_1)\frac{\partial x_o}{\partial\xi} + (B[K - B^2]u_1 + v_1)\frac{\partial r_o}{\partial\xi} = 0 \tag{3.13}$$

$$\frac{\partial}{\partial\eta}(x_1 - B\,r_1) + (u_1 + B\,v_1)\frac{\partial x_o}{\partial\eta} - (B[K - B^2]u_1 - v_1)\frac{\partial r_o}{\partial\eta} = 0 \tag{3.14}$$

It is readily verified using (3.10) - (3.12) that the solutions of (3.13) and (3.14) can be written

$$x_1 + B\,r_1 + M^2\phi_1 + K\,r_o(B\,u_1 + v_1) = \lambda_3(\eta) \tag{3.15}$$

$$x_1 - B\,r_1 + M^2\phi_1 - K\,r_o(B\,u_1 - v_1) = \lambda_4(\xi) \tag{3.16}$$

The discovery of the explicit solution first given in [12] makes possible considerable simplifications of the results.

Becuase of the condition that disturbances vanish upstream of the body so that the Mach lines then become those of the undisturbed flow we select $\lambda_3 \equiv 0$. Also, the solutions for u_2 and v_2 turn out to be particularly simple with the choice $\lambda_4 \equiv 0$. Thus, we have

$$x_1 = -K\,r_o\,v_1(x_o, r_o) - M^2\phi_1(x_o, r_o) \tag{3.17}$$

$$r_1 = -K\,r_o\,u_1(x_o, r_o) \tag{3.18}$$

Now we return to (2.7) and (2.8), Substitution of (3.17) and (3.18) and retention of the next order terms yields

$$\left[\frac{\partial}{\partial\xi}(B\,u_2 + v_2) + \frac{v_2}{r_o}\frac{\partial r_o}{\partial\xi}\right] + \left[B(K-B^2)u_1 - v_1\right]\frac{\partial u_1}{\partial\xi} + \\ + (u_1 + B\,v_1)\frac{\partial v_1}{\partial\xi} + (u_1 + B\,v_1)\frac{v_1}{r_o}\frac{\partial r_o}{\partial\xi} - K\,v_1\frac{\partial u_1}{\partial\xi} = 0 \tag{3.19}$$

$$\left[\frac{\partial}{\partial\eta}(B\,u_2 - v_2) - \frac{v_2}{r_o}\frac{\partial r_o}{\partial\eta}\right] + \left[B(K-B^2)u_1 + v_1\right]\frac{\partial u_1}{\partial\eta} - \\ - (u_1 - B\,v_1)\frac{\partial v_1}{\partial\eta} - (u_1 - B\,v_1)\frac{v_1}{r_o}\frac{\partial r_o}{\partial\eta} + K\,v_1\frac{\partial u_1}{\partial\eta} = 0 \tag{3.20}$$

With the aid of (3.6) and (3.7) and with introduction of the new dependent variables

$$u_o = u_1 + u_2 - M^2 u_1^2 \tag{3.21}$$

$$v_o = v_1 + v_2 - (M^2 + K) u_1 v_1 \tag{3.22}$$

we find that (3.19) and (3.20) may be written in the following simple way:

$$\frac{\partial}{\partial \xi} (B u_o + v_o) + \frac{v_o}{r_o} \frac{\partial r_o}{\partial \xi} = 0 \tag{3.23}$$

$$\frac{\partial}{\partial \eta} (B u_o - v_o) - \frac{v_o}{r_o} \frac{\partial r_o}{\partial \eta} = 0 \tag{3.24}$$

Thus, a comparison with (3.6) and (3.7) shows that u_o and v_o are obtained from the solution of the linearized equation

$$- B^2 \frac{\partial^2 \phi_o}{\partial x_o^2} + \frac{1}{r_o} \frac{\partial \phi_o}{\partial r_o} + \frac{\partial^2 \phi_o}{\partial r_o^2} = 0 \tag{3.25}$$

with $u_o = \partial \phi_o / \partial x_o$, $v_o = \partial \phi_o / \partial r_o$. Therefore, the second-order solution can be expressed directly in terms of the solution of the linear equation (3.25) by

$$u = (1 + M^2 u_o) u_o \tag{3.26}$$

$$v = (1 + [K + M^2] u_o) v_o \tag{3.27}$$

where u_o and v_o are considered functions of x_o and r_o , and x and r are given by

$$x = x_o - M^2 \phi_o - K r_o v_o \tag{3.28}$$

$$r = r_o (1 - K u_o) \tag{3.29}$$

One can also readily show that the velocity potential to second

order is given by

$$\phi = \phi_o - K\, r_o\, u_o\, v_o \tag{3.30}$$

Next, we shall extend this solution to the region close to the body, $r = O(\epsilon)$, by making use of Van Dyke's [7] solution. Through a regular expansion procedure Van Dyke found the solution for the velocity potential $\phi = \phi_1 + \phi_2$ to be

$$\phi_2 = u_1(M^2\phi_1 + K r v_1) - \frac{M^2}{4} r v_1^3 \tag{3.31}$$

from which the following velocity components can be obtained:

$$\begin{aligned} u = u_1 + u_2 &= u_1\left(1 + M^2 u_1 + K r \frac{\partial u_1}{\partial r}\right) + \frac{\partial u_1}{\partial x}(M^2 \phi_1 + K r v_1) - \\ &- \frac{3}{4} M^2 r \frac{\partial u_1}{\partial r} v_1^2 \end{aligned} \tag{3.32}$$

$$\begin{aligned} v = v_1 + v_2 &= v_1\left[1 + (M^2 + K) u_1 + K r \frac{\partial v_1}{\partial r}\right] + \frac{\partial v_1}{\partial x}(M^2\phi_1 + K r v_1) - \\ &- \frac{M^2}{4}\left(v_1^3 + 3 r v_1^2 \frac{\partial v_1}{\partial r}\right) \end{aligned} \tag{3.33}$$

These are to be compared to the velocities obtained by expanding the uniformly valid solution for $r = \text{ord}\,(1)$ obtained above.

Setting $u_1(x_o, r_o) = u_1 + (x - x_o)\frac{\partial u_1}{\partial x} + (r - r_o)\frac{\partial u_1}{\partial r} + ,\ r_o = r + \ldots,$ etc., we obtain

$$u = u_1\left(1 + M^2 u_1 + K r \frac{\partial u_1}{\partial r}\right) + \frac{\partial u_1}{\partial x}(M^2\phi_1 + K r v_1) \tag{3.34}$$

$$v = v_1\left[1 + (M^2 + K) u_1 + K r \frac{\partial v_1}{\partial r}\right] + \frac{\partial v_1}{\partial x}(M^2 \phi_1 + K r v_1) \tag{3.35}$$

which differ from (3.32), (3.33) only in that the triple product terms are missing. Van Dyke [7] pointed out that these will be of the same order as the quadratic ones near a slender body for $r = O(\epsilon)$. For

$r = \text{ord}\ (1)$, however, they are negligible and of the same order as contributions from other terms already neglected in the differential equation. It is possible to make the solution (3.26) - (3.29) uniformly valid also for $r = O(\epsilon)$ by modifying it as follows:

$$x = x_o - K\,r_o\,v_o - M^2\phi_o + \frac{3}{4} M^2 B r_o\, v_o{}^2 + \frac{1}{2}\lambda_5(\xi) \tag{3.36}$$

$$r = r_o(1 - K\,u_o) - \frac{3}{4} M^2\, r_o\, v_o{}^2 - \frac{1}{2B}\lambda_5(\xi) \tag{3.37}$$

$$v = [1 + (K+M)\,u_o - \frac{1}{4} M^2\, v_o{}^2]\, v_o \tag{3.38}$$

The expression for u_o is the same as before. The additional terms involving the as yet undetermined function λ_5, and the one with $v_o{}^2$ in (3.36), are everywhere of higher order and incorporated to avoid discontinuous behaviour of x when u_o and v_o have discontinuities at a body slope discontinuity. The term r has been included in order to eliminate the spatial derivative terms in the triple product contribution to u and v given by the last terms in (3.32) and (3.33). It becomes comparable to the previously derived term $-K\,r_o u_o$ only for $r = O(\epsilon)$ and is therefore at most of order ϵ^3. In constructing the new term we have freely taken advantage of the slender-body behaviour of v_o. Thus, for example, for $r_o = O(\epsilon)$, $\partial v_1/\partial r \simeq \partial v_o/\partial r \simeq -v_o/r_o$.

To determine λ_5, let the body contour in the transformed plane be defined by

$$r_o = R_o(x_o) \tag{3.39}$$

By choosing

$$\lambda_5 = -2BK\,u_B(\xi)\,R_o(x_{oB}) - \frac{3}{2} M^2\, B\, v_B^2\, R_o(x_{oB}) \tag{3.40}$$

where $u_B = u_o(x_{oB},\, r_{oB})$, $v_B = v_o(x_{oB},\, r_{oB})$ and x_{oB}, r_{oB} are the

coordinates for the intersection of the particular characteristic ξ = const. with the body contour (see figure 3.1) we can ascertain that both x_o and r_o become continuous at points of body slope discontinuities.

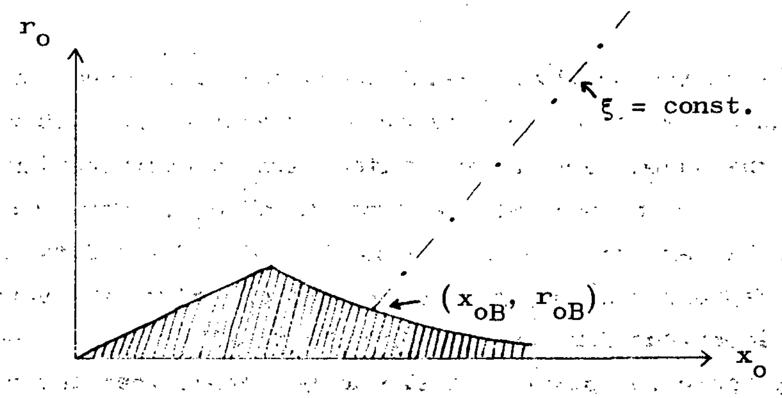


Figure 3.1

This follows, because in the immediate neighbourhood behind the body corner the flow must behave locally like a two-dimensional one with the velocity component jumps to lowest order satisfying $\Delta u_B = -\Delta v_B/B = -\Delta R'/B$, where $\Delta R'$ is the slope discontinuity. From the expressions (3.36) and (3.37) it then follows that a continuous body contour in the x_o, r_o plane leads to a continuous contour in the physical plane with

$$r = R = R_o[x_{oB}(x)] \tag{3.41}$$

and x given implicitly by

$$x = x_{oB} - M^2\phi_B - KR_o(v_B + B\,u_B) \tag{3.42}$$

(Continuity in x follows since the combination $v_o + Bu_o$ is continuous, as well as ϕ_B .) Thus, a solution, uniformly valid to second order everywhere for $r = O(1)$, is given by

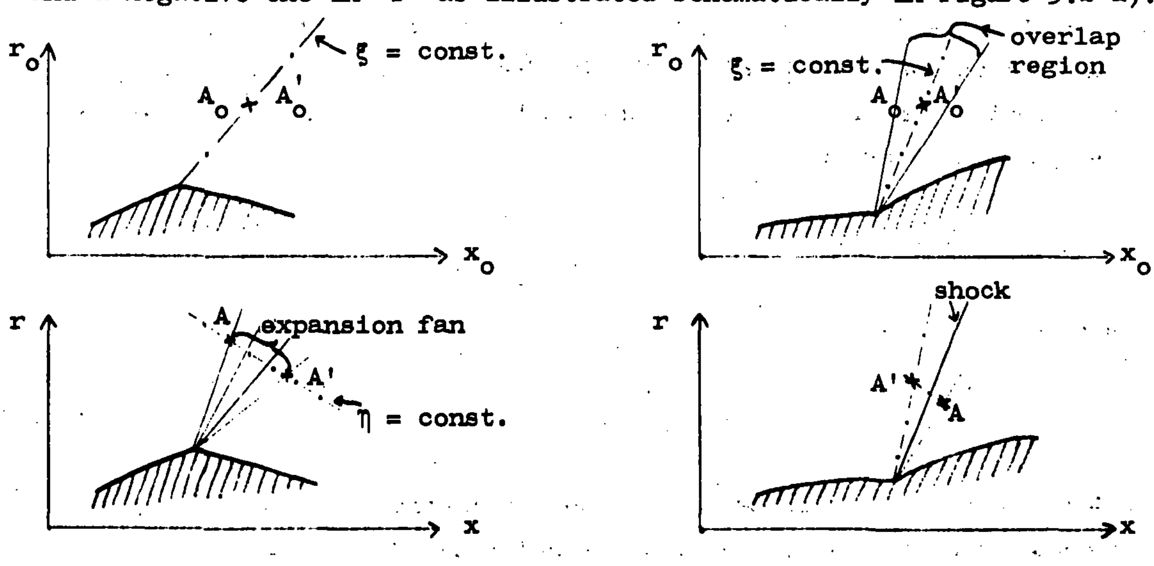
$$u = u_o(1 + M^2u_o) \tag{3.43}$$

$$v = v_o[1 + (K + M^2)\,u_o - \frac{M^2}{4}\,v_o{}^2] \tag{3.44}$$

$$x = x_o - K\,r_o\,v_o - M^2\phi_o - BKR_o\,u_B + \frac{3BM^2}{4}\,(r_o\,v_o^{\,2} - R_o\,v_B^{\,2}) \tag{3.45}$$

$$r = r_o(1 - K\,u_o) + K\,R_o\,u_B - \frac{3M^2}{4}(r_o\,v_o^{\,2} - R_o\,v_B^{\,2}) \tag{3.46}$$

A comparison with the previous result shows that the added terms give contributions which are of higher order everywhere with the exception of the added term in v, which gives a contribution of the same order near the body as the term $(K-M^2)u_o$. For $r = \mathrm{ord}(1)$ it becomes of higher order and may be neglected. The additional terms in the expressions for x and r are of order ϵ^3 everywhere and are only important for representing the details of a non-smooth body in the transformed plane. It should be noted, that for a body contour with slope discontinuities, the transformations $x = x(x_o, r_o)$ and $r(x_o, r_o)$ as given by (3.45) and (3.46) will be discontinuous outside the body along the downstream characteristic from the corner. For a convex corner (expansion) there is a negative jump in v_o and a corresponding positive one in u_o giving a positive jump in x and a negative one in r as illustrated schematically in figure 3.2 a).



a) Convex corner (expansion)

b) Concave corner (compression)

Figure 3.2. Coordinate transformation near discontinuities.

A double point A_o A_o', on the upstream and downstream sides, respectively, of the characteristic ξ = const. from the corner thus becomes transformed into a line along η = const. as shown. The gap between A and A' in the physical plane must of course be filled with an expansion fan, as is well known from the two-dimensional theory. In the present problem the actual variation of flow quantities across the fan can be determined by considering the sharp corner as the limiting case of a rounded one as the radius of curvature tends to zero. For a concave corner, on the other hand, the downstream point A_o' is transformed to an upstream one in the physical plane (see figure 3.2 b)) and a problem of non-uniqueness thus arises, since one and the same region in the physical plane might correspond to two regions in the transformed one. This difficulty is to be resolved by the insertion of a shock wave to remove the overlap region in the transformed plane. Such regions may of course also appear even for a body without sharp concave corners, and the problem of determining shock wave locations will therefore require a more general treatment given in the following section.

With the uniformly valid solution near the body thus derived we may proceed to consider the boundary condition on the body surface. By introducing the second-order solution (3.43) - (3.46) into the tangency condition (3.2) we find upon neglecting higher-order terms,

$$v_o[1 + (K + M^2)u_o - \frac{M^2}{4} v_o^2] = (1 + u_o)R'(x) \tag{3.47}$$

which, together with the initial condition of zero disturbances upstream of the body, provides the boundary condition for the linear equation (3.25). As the transformed body contour is continuous, the problem of finding the transformed body contour in the x_o,r_o-plane can be solved by iteration starting from the ordinary first-order solution with the linearized boundary condition

$$v_1(x\ ,R(x)) = R'(x) \tag{3.48}$$

It remains to be studied the behaviour of the solution in the far field, $r = \text{ord}\ (\epsilon^4)$, $\xi = 0\ (1)$. In Appendix D it is shown that by assuming

$$u = \epsilon^4 U_1 + \epsilon^8 U_2 + \ldots$$

$$v = \epsilon^4 V_1 + \epsilon^8 V_2 + \ldots$$

$$x - Br \equiv \chi = \chi_o + \epsilon^4 \chi_1 + \ldots$$

$$\epsilon^4 \cdot 2Br \equiv H = H_o + \epsilon^4 H_1 + \ldots$$

where U_k, V_k, χ_k and H_k are functions of ξ and η, all of order unity, and $H_o = \text{ord}\,(1)$, one finds that the solution may be written

$$u = -\frac{F(\xi)}{\sqrt{2Br}} + \frac{1}{4}\frac{1}{(2Br)^{3/2}}\int_0^{\xi} F(\xi')d\xi' + (M^2 - \frac{K}{2})\frac{F^2(\xi)}{2Br} + O(\epsilon^{12}) \qquad (3.49)$$

$$v = -Bu + \frac{B}{(2Br)^{3/2}}\int_0^{\xi} F(\xi')d\xi' - BK\frac{F^2(\xi)}{2Br} + O(\epsilon^{12}) \qquad (3.50)$$

$$x - Br = \xi - K\sqrt{2Br}\,F(\xi) + \frac{M^2 - \frac{K}{4}}{\sqrt{2Br}}\int_0^{\xi} F(\xi')d\xi' + \qquad (3.51)$$

$$+ \; L\,F^2(\xi)\ \ln Br + \lambda(\xi) + O(\epsilon^8)$$

where $\quad L = \frac{K}{2}\,(K + \frac{3}{2}M^2 - \frac{2K}{M^2})$

From the mid-field solution expanded in the same manner one finds, using the far-field expansions for u_o, v_o and ϕ_o given in

Appendix C, and noting that from (3.37) $r_o \cong r + Kru_o + O(\epsilon^3)$, the same expressions for u and v as above, but that (3.51) is replaced by

$$x - Br = \xi - K\sqrt{2Br}\,F(\xi) + \frac{M^2 - \frac{K}{4}}{\sqrt{2Br}} \int_0^{\xi} F(\xi')d\xi' + \\ + \frac{3}{4} M^2 B^2 F^2(\xi) + \lambda_5 \qquad (3.52)$$

The last two terms may be absorbed in the function $\lambda(\xi)$ of (3.51). However, the term proportional to $\ln Br$ which is the result of higher-order cumulative effects in the characteristics in (3.51) has no correspondence in (3.52). We thus need to complement the midfield solution with terms which behave like $LF^2 \ln Br$ in the far field, but which will only give higher-order contributions near the body. The added terms, however, must not lead to discontinuities in the transform of the body contour. This can be achieved by replacing (3.36) and (3.37) by

$$x = x_o - K r_o v_o - M^2\phi_o + \frac{3}{4} M^2 B r_o v_o^2 + \\ + L B r_o (\ln B r_o) u_o^2 + \frac{1}{2}\lambda_6 \qquad (3.53)$$

$$r = r_o(1 - K u_o) - \frac{3}{4} M^2 r_o v_o^2 - L r_o \ln(Br_o) u_o^2 - \frac{1}{2B}\lambda_6 \qquad (3.54)$$

where

$$\lambda_6 = -2BK u_B(\xi) R_o(x_{oB}) - \frac{3}{2} B v_B^2 R_o(x_{oB}) - \\ - 2LBR_o \ln(BR_o) u_B^2 \qquad (3.55)$$

The transformation (3.53) - (3.55) together with (3.43) and (3.44) provides a solution which is uniformly valid to second order everywhere. The new terms in (3.53) and (3.54) give contributions to x and r which are at most of order ϵ^4. In practice they are likely to be unimportant since they lead to contributions in the mid field of order ϵ^8 in the far field. Finally, it should also be remarked

that a uniformly valid solution can be constructed in many different ways and the one given here was selected with computational simplicity in mind.

4. DETERMINATION OF SHOCK WAVES FOR AXISYMMETRIC FLOW

At a shock wave, the following conditions hold (see, e.g., [1], p. 274):

$$\vec{q}^{(1)} \times \vec{q}^{(2)} = \mu^2 \hat{q}^2 + (1-\mu^2)(\vec{q}^{(1)} \times \vec{n})^2 \tag{4.1}$$

$$\vec{q}^{(1)} \times \vec{n} = \vec{q}^{(2)} \times \vec{n} \tag{4.2}$$

where $\vec{q}^{(1)}$ and $\vec{q}^{(2)}$ are the velocity vectors ahead and behind the shock, respectively, $\mu^2 = (\gamma-1)/(\gamma+1)$, $\hat{q}^2 = U^2 + V^2 + W^2 + 2c^2/(\gamma-1)$, c the local velocity of sound, and $\vec{n}$ is the unit normal to the shock. Let the shock in the axisymmetric case be defined by $x = \psi(r)$. Then (4.1) gives

$$\left(\frac{d\psi}{dr}\right)^2 = B^2 + \frac{K\,B^2}{M^2}\left[1 + \left(\frac{d\psi}{dr}\right)^2\right]\left(u^{(1)} + u^{(2)} + u^{(1)}u^{(2)} + \right.$$
$$\left. + v^{(1)}v^{(2)} - 2M^2 \frac{d\psi}{dr}\left(u^{(1)}\frac{d\psi}{dr} + v^{(1)}\right) - M^2\left(u^{(1)}\frac{d\psi}{dr} + v^{(1)}\right)^2 \tag{4.3}$$

Expanding this result for small values of the perturbation velocity components and retaining only terms of zeroth and first order in consistency with the second-order theory for the mid-field, we find

$$\frac{d\psi}{dr} = B\left[1 + \frac{K}{2}\left(u^{(1)} + u^{(2)}\right) - \frac{M^2}{B}\left(B\,u^{(1)} + v^{(1)}\right) + \dots\right] \tag{4.4}$$

The second shock condition, (4.2), simply expresses that the tangential velocity component is continuous across the shock. Thus,

$$\left(u^{(2)} - u^{(1)}\right)\frac{d\psi}{dr} + v^{(2)} - v^{(1)} = 0 \tag{4.5}$$

Combining this with (4.4) we find that, to lowest order,

$$\frac{v^{(2)} - v^{(1)}}{u^{(2)} - u^{(1)}} = \frac{\Delta v}{\Delta u} = - B + \ldots \tag{4.6}$$

It can readily be shown that the condition (4.2) of continuous tangential component, to the order considered, is equivalent to the requirement that the potential be continuous across the shock.

Considering first the mid-field region, r = ord (1), we obtain from (3.30) and the requirement that the potential is continuous across the shock

$$\Delta\phi = \Delta\phi_o - K\,\Delta(r_o\, u_o\, v_o) = 0 \tag{4.7}$$

Now, from (4.6) it follows that, to lowest order,

$$\Delta(v_o + Bu_o) = 0 \tag{4.8}$$

across the shock. Therefore, making use of the identity

$$u_o\, v_o = - \frac{1}{4B}\left[(v_o - Bu_o)^2 - (v_o + Bu_o)^2\right]$$

we may rewrite (4.7) as follows

$$\Delta\phi_o = - \frac{K}{2}\left\{\Delta[F^2(\xi,\eta)] - \frac{1}{2}\left(v_o^{(1)} + Bu_o^{(1)}\right)^2 \Delta r_o\right\} \tag{4.9}$$

where

$$F(\xi,\eta) = \frac{1}{2B}\sqrt{2Br_o}\,(v_o - Bu_o) \equiv \frac{1}{2B}\sqrt{\eta - \xi}\,(v_o - Bu_o) \tag{4.10}$$

may be termed the "generalized Whitham F-function" as it tends to the ordinary Whitham F-function for $\eta \to \infty$. The transformations (3.28) and (3.29) give

$$x + Br = \eta - \frac{K}{2B}(\eta - \xi)(v_o + Bu_o) - M^2\phi_o \tag{4.11}$$

$$x - Br = \xi - \frac{K}{2B}(\eta - \xi)(v_o - Bu_o) - M^2\phi_o \tag{4.12}$$

Since both Δx and Δr are zero across the shock these relations yield with the aid of (4.8)

$$\Delta\eta\left[1 + \frac{K}{2B}\left(v_o^{(1)} + B\,u_o^{(1)}\right)\right] = \frac{K}{2B}\Delta\xi\left(v_o^{(1)} + B\,u_o^{(1)}\right) + M^2\,\Delta\phi_o \tag{4.13}$$

$$\Delta\xi = \frac{K}{2B}\left[\left(\eta^{(2)} - \xi^{(2)}\right)\left(v_o^{(2)} - B\,u_o^{(2)}\right) - \left(\eta^{(1)} - \xi^{(1)}\right)\left(v_o^{(1)} - B\,u_o^{(1)}\right)\right] + M^2\,\Delta\phi_o \tag{4.14}$$

From (4.13) it follows that $\Delta\eta$ is at most of order u_o, i.e. $O(\epsilon^2)$, so that to lowest order the shock wave transition takes place along lines of constant η. The term involving $\Delta\phi_o$ is of order $\Delta\xi$ times u_o and is therefore only comparable to the first bracket of (4.14) for the near field $r_o = (\epsilon)$, in which case both are negligible. From (4.13) and (4.14) it also follows that $\Delta r_o/r_o^{(1)} = \Delta(\eta-\xi)/\left(\eta^{(1)} - \xi^{(1)}\right) = O(\epsilon^2)$ so that the change in r_o may be neglected compared to r_o itself. It therefore follows, that to within the approximation considered (4.9) and (4.14) may be simplified to

$$\int_{\xi^{(1)}}^{\xi^{(2)}} F(\xi,\eta)\,d\xi = \frac{1}{2}K\sqrt{2B\,r_o}\left[\left(F^{(2)}\right)^2 - \left(F^{(1)}\right)^2\right] \tag{4.15}$$

$$\xi^{(2)} - \xi^{(1)} = K\sqrt{2B\,r_o}\left(F^{(2)} - F^{(1)}\right) \tag{4.16}$$

The left-hand side of (4.15) follows from the definition (4.10), since

$$\frac{1}{2B}\left(v_o - B u_o\right) = -\frac{\partial \phi_o}{\partial \xi} \tag{4.17}$$

The relations (4.15) and (4.16) express Whitham's "rule of equal areas" [8] since by substituting $F^{(2)} - F^{(1)}$ from (4.16) into (4.15) one obtains

$$\int_{\xi^{(1)}}^{\xi^{(2)}} F(\xi,\eta)\, d\xi = \frac{1}{2}\left(\xi^{(2)} - \xi^{(1)}\right)\left(F^{(2)} + F^{(1)}\right) \tag{4.18}$$

Thus, by aid of (4.16) and (4.18) one can use the same graphical technique as commonly employed in the first-order far-field solution to determine the shock wave location in the mid field, provided the definition (4.10) is substituted for the F-function.

Consider next the near- and the far-field regions. For the near field region, $r = O(\epsilon)$, some additional triple-product terms were required both in the expressions for x and r and in the v-velocity component because of the slender-body behaviour near a smooth body. However, it is easily demonstrated that for a smooth body shocks can only occur for $(\xi/r) = O(\epsilon^4)$ in which case the triple-product term is negligible. For a body with slope discontinuities, shocks will occur also for $r = O(\epsilon)$ $\xi = O(1)$, but then the flow will behave locally near the corner like a two-dimensional one and slender-body triple product terms will be negligible for the determination of the shock location. Thus, the mid-field procedure for determining the shock location will be valid to the required order of accuracy for the near field, as well. For the far field, it was demonstrated in the previous section that the mid-field solution gave a uniformly valid solution to lowest order. Thus, the mid-field procedure will give shock locations correct to order unity with errors in location of order ϵ^4 and shock strength errors of order ϵ^8. Since such errors are likely to be unimportant in practice, the added complication of including the higher-order terms in the shock wave location for large distances does not seem worth while.

5. SECOND-ORDER THEORY FOR THE FAR FIELD IN A THREE-DIMENSIONAL FLOW

We consider the flow around a simple airplane-like form (wing-body combination) located in the half-plane $x \geq 0$ with its (pointed) nose at $x = r = 0$ and the fuselage mainly along the positive x-axis. A suitable perturbation parameter ϵ is here the thickness ratio of the least slender of equivalent bodies of revolution in the supersonic area-rule sense.

With Section 3 in mind, a suitable procedure might seem to be to first carry out an expansion in the mid field and then extend it to the near- and far-field regions. However, it is found that the mid-field analysis employing characteristic variables for the general three-dimensional case does not lead to a simple explicit solution as in the axisymmetric case. This indicates that the concept of characteristic variables is not of the same usefulness near the body as in the axisymmetric case. Therefore, we will restrict ourselves to the study of the far field. Following Section 3, we define the far-field domain by

$$\begin{aligned} x - Br &= O(1) \\ 2\,Br &= \operatorname{ord} \epsilon^{-4} \end{aligned}$$

An expansion similar to the one used in the axisymmetric case is assumed

$$\begin{cases} u = \epsilon^4 U_1 + \epsilon^8 U_2 + \dots \\ v = \epsilon^4 V_1 + \epsilon^8 V_2 + \dots \\ w = \epsilon^4 W_1 + \epsilon^8 W_2 + \dots \\ x - Br \equiv \chi = \chi_o + \epsilon^4 \chi_1 + \dots \\ 2B\epsilon^4 r \equiv H = H_o + \epsilon^4 H_1 + \dots \\ \theta = \theta_o + \epsilon^4 \theta_1 + \dots \end{cases} \tag{5.1}$$

where U_k, V_k, W_k, χ_k, H_k and θ_k are considered functions of order unity of the characteristic variables ξ, η and ζ. The expressions (5.1) could then be substituted into the equations of motion expressed in characteristic form and a sequence of equations

generated thereby. However, to obtain the lowest order terms it is actually simpler to use (2.1) - (2.4) employing the new variables χ and H. In these, the irrotationality condition (2.2) becomes

$$\frac{\partial V}{\partial \chi} = 2B\,\epsilon^4 \frac{\partial U}{\partial H} - B\frac{\partial U}{\partial \chi} \tag{5.2}$$

from which it follows that, to lowest order,

$$V_1 = -B\,U_1 \tag{5.3}$$

Also, (2.3) gives that

$$W_1 = 0 \tag{5.4}$$

With the aid of these results, we obtain through substitution of the series (5.1) into (2.1) when higher-order terms are omitted

$$\frac{K}{2} U_1 \frac{\partial U_1}{\partial \chi_o} + \frac{\partial U_1}{\partial H_o} + \frac{1}{2H_o} U_1 = 0 \tag{5.5}$$

This equation is most easily solved by means of the Monge theory. It was shown in [10] that the solution can be given in parametric form as follows

$$U_1 = -\frac{\mathcal{F}_1(\xi,\theta_o)}{\sqrt{H_o}} \tag{5.6}$$

$$\chi_o = \xi - K\sqrt{H_o}\,\mathcal{F}_1(\xi,\theta_o) \tag{5.7}$$

By identifying the parameter with ξ we get a simple and obvious relationship between $\mathcal{F}_1$ and the Whitham F-function. Further, as in the axisymmetric case H_o must be a function of η, alone, and we may set

$$H_o = \eta \tag{5.8}$$

$$\theta_o = \zeta \tag{5.9}$$

Next, we proceed by substitution of the series (5.1) into (2.9) - (2.14). Using the above results for the lower-order terms we obtain the following set of equations to determine the next-order terms:

$$\frac{\partial}{\partial \xi}\left(\chi_o + H_1\right) = 0 \tag{5.10}$$

$$\frac{\partial \chi_1}{\partial \eta} = \frac{K}{4B^2}\left[3K - M^2 - \frac{4K}{M^2}\right]\frac{\partial H_o}{\partial \eta} V_1{}^2 - \frac{K}{2B}\frac{\partial H_1}{\partial \eta} V_1 - \tag{5.11}$$

$$-\frac{1}{2B}\frac{\partial H_o}{\partial \eta}\left(M^2 V_2 + [M^2 - K]\, B\, U_2\right) - \underline{\frac{K^2}{B^2}\left(\frac{\partial V_1}{\partial \zeta}\right)^2} - \underline{\frac{K}{B} H_o \frac{\partial \theta_1}{\partial H_o}\frac{\partial V_1}{\partial \zeta}} \tag{5.12}$$

$$\frac{\partial \theta_1}{\partial \eta} = -\underline{\frac{2K}{B}\frac{1}{H_o}\frac{\partial V_1}{\partial \zeta}\frac{\partial H_o}{\partial \eta}} \tag{5.13}$$

$$\frac{\partial}{\partial \xi}\left(B U_2 + V_2\right) = -\frac{K}{2B}\frac{\partial V_1{}^2}{\partial \xi} - \frac{V_1}{H_o}\frac{\partial H_1}{\partial \xi} \tag{5.14}$$

$$\frac{\partial}{\partial \eta}\left(B U_2 - V_2\right) = -\frac{1}{H_o}\frac{\partial H_o}{\partial \eta}\frac{\left(M^2 - \frac{K}{2}\right)}{B} V_1{}^2 + \frac{V_1}{H_o}\frac{\partial H_1}{\partial \eta} - \frac{V_1\, H_1}{H_o{}^2} +$$

$$+\frac{V_2}{H_o}\frac{\partial H_o}{\partial \eta} + \underline{\frac{1}{H_o}\frac{\partial H_o}{\partial \eta}\frac{\partial W_2}{\partial \zeta}} - \underline{\frac{2K}{BH_o}\left(\frac{\partial V_1}{\partial \zeta}\right)^2\frac{\partial H_o}{\partial \eta}} - \underline{2\,\frac{\partial \theta_1}{\partial \eta}\frac{\partial V_1}{\partial \zeta}} \tag{5.15}$$

$$\frac{\partial W_2}{\partial \xi} = -\underline{\frac{2}{H_o}\frac{\partial V_1}{\partial \zeta}\frac{\partial H_o}{\partial \eta}} \tag{5.16}$$

These equations differ from those in the axisymmetric case only by the underlined terms, and their integration will proceed in very much the same manner as in Appendix D. The final results read

$$H_1 = -\chi_o + h_1(\eta, \zeta) \tag{5.17}$$

$$\left\{\begin{aligned} \chi_1 &= \frac{M^2 - \frac{K}{4}}{B}\int_0^{\xi} V_1(\xi', \eta, \zeta)\,d\xi' + \frac{K}{2B^2}\left(K + \frac{3}{2}M^2 - \frac{2K}{M^2}\right) V_1{}^2 H_o \ln H_o - \\ &- \frac{K}{2B} V_1\, H_1 - K\sqrt{H_o}\,\mathcal{F}_2 + \ell_1(\xi,\zeta) - \underline{\frac{K}{B}\int_0^{\xi}\frac{\partial^2 V_1}{\partial \zeta^2}(\xi', \eta, \zeta)\,d\xi'} \end{aligned}\right. \tag{5.18}$$

$$\underline{\theta_1 = \frac{4K}{B}\frac{\partial V_1}{\partial \zeta} + \vartheta_1(\xi,\zeta)} \tag{5.19}$$

$$B\,U_2 + V_2 = -\frac{K}{B}V_1^{\,2} + \frac{1}{H_o}\int_o^{\xi} V_1(\xi',\eta,\zeta)d\xi' \tag{5.20}$$

$$\left\{\begin{aligned} V_2 = &-\frac{M^2+\frac{K}{2}}{B}V_1^{\,2} - \frac{H_1\,V_1}{2H_o} + \frac{3}{4H_o}\int_o^{\xi} V_1(\xi',\eta,\zeta)d\xi' + \\ &+ \frac{B\,\mathcal{F}_2(\xi,\zeta)}{\sqrt{H_o}} + \underline{\frac{2K}{B}\left(\frac{\partial V_1}{\partial\zeta}\right)^2} - \underline{\frac{1}{H_o}\int_o^{\xi}\frac{\partial^2 V_1}{\partial\zeta^2}(\xi',\eta,\zeta)d\xi'} \end{aligned}\right. \tag{5.21}$$

$$\underline{W_2 = -\frac{2}{H_o}\int_o^{\xi}\frac{\partial V_1}{\partial\zeta}(\xi',\eta,\zeta)d\xi'} \tag{5.22}$$

The underlined terms are the new ones which appear due to departure from axisymmetry. A comparison with the axisymmetric solution therefore suggests that we may write the far-field solution for the three-dimensional case in the following simplified form

$$u = u^{(a)} - 2K\left(\frac{\partial u_o}{\partial\zeta}\right)^2 \tag{5.23}$$

$$v = v^{(a)} + 2KB\left(\frac{\partial u_o}{\partial\zeta}\right)^2 \tag{5.24}$$

$$w = w_o \tag{5.25}$$

$$x = x^{(a)} + \frac{K\,r_o}{2}\frac{\partial w_o}{\partial\zeta} \tag{5.26}$$

$$r = r^{(a)} - \frac{K}{2B}\,r_o\frac{\partial w_o}{\partial\zeta} \tag{5.27}$$

$$\theta = \zeta - 4K\frac{\partial u_o}{\partial\zeta} \tag{5.28}$$

where superscript (a) means the same expression as in the axisymmetric case and u_o, v_o and w_o are obtained from the solution of

$$-B^2 \frac{\partial^2 \phi_o}{\partial x_o^2} + \frac{1}{r_o}\frac{\partial \phi_o}{\partial r_o} + \frac{\partial^2 \phi_o}{\partial r_o^2} + \frac{1}{r_o^2}\frac{\partial^2 \phi_o}{\partial \zeta^2} = 0 \tag{5.29}$$

The terms in the expressions for u and v containing derivatives with respect to the angular variable ζ can be eliminated by making the further transformation

$$\theta_o = \zeta - 2K \frac{\partial u_o}{\partial \zeta} \tag{5.30}$$

which leads to the final expressions

$$u = u^{(a)} \tag{5.31}$$

$$v = v^{(a)} \tag{5.32}$$

$$w = w_o \tag{5.33}$$

$$x = x^{(a)} + \frac{Kr_o}{2}\frac{\partial w_o}{\partial \theta_o} + 2BK^2 r_o\left(\frac{\partial u_o}{\partial \theta_o}\right)^2 \tag{5.34}$$

$$r = r^{(a)} - \frac{Kr_o}{2B}\frac{\partial w_o}{\partial \theta_o} - 2K^2 r_o\left(\frac{\partial u_o}{\partial \theta_o}\right)^2 \tag{5.35}$$

$$\theta = \theta_o - 2K \frac{\partial u_o}{\partial \theta_o} \tag{5.36}$$

Here, u_o, v_o and w_o are now solutions of (5.29) with ζ replaced by θ_o. In considering the shocks in the far field one notices that to lowest order the solution has the same functional behaviour as the axisymmetric one, except that θ_o appears as additional parameter in the F-function. One can therefore use the same method as in the axisymmetric case to determine the shock wave location to first order.

6. EVALUATION OF THE F-FUNCTION FROM WIND-TUNNEL EXPERIMENTS

A new method to determine the F-function from wind tunnel measurements, was proposed in [14]. In it, flow deflections are measured instead of pressures as in previously used methods. Thereby, considerably improved accuracy is made possible, particularly at high Mach numbers, since flow deflections are much easier to measure accurately than pressures. The relationship between flow deflections and the F-function becomes simple only in the far field, so that the flow deflections must be measured sufficiently far away for the flow to have approached its far field behaviour, but yet close enough where the deflections still reasonably large to allow good measurement accuracy. Inspection of the linear solution for axisymmetric flow reveals that its far-field behaviour is attained when Br is large. For slender configurations the far field character may be dominant already for values of Br greater than 3, or sometimes even less. This makes the new method particularly useful for high Mach numbers for which the far field may be approached already for $r = \mathrm{ord}(1)$.

The new experimental technique utilizes measurements along a cylindrical surface $r = \text{const.}$ of flow deflections σ and τ in the radial and azimuthal directions, respectively. From figure 6.1 one obtains

$$\left.\begin{aligned} \mathrm{tg}\,\sigma &= \frac{v}{1+u} \\ \mathrm{tg}\,\tau &= \frac{w}{1+u} \end{aligned}\right\} \qquad (6.1)$$

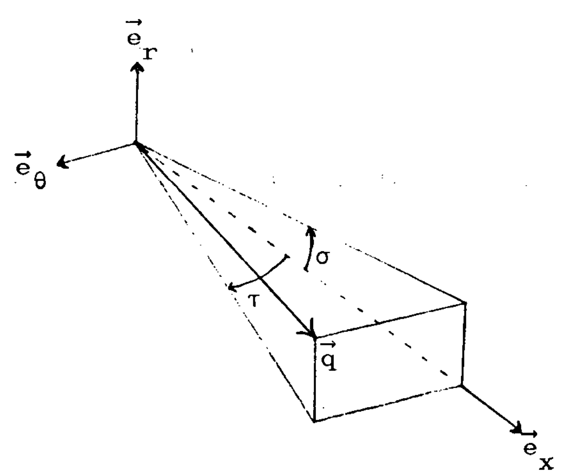


Figure 6.1

In order to make direct use of the results from the previous section to relate these to the F-function we shall tentatively assume that $Br = \text{ord}(\epsilon^{-4})$ so that these results are directly applicable. Through expansion for small σ and τ and making use of the far-field approximation for u, v and w in terms of u_o, v_o and w_o one can derive expressions for u_o, v_o, w_o in terms of σ and τ (see Appendix E). These, in turn, can be used in the expressions relating the velocities to the F-function producing the following relations:

$$F(\xi,\theta_o) = \sqrt{\frac{2r}{B}}\,\sigma(x,\theta) - \frac{3}{8B}\sqrt{\frac{2}{Br}}\int_0^x \sigma(x',\theta)dx' - \frac{1}{\sqrt{2Br}}\frac{\partial\tau(x,\theta)}{\partial\theta} + \underline{(B+\frac{K}{8B})\sqrt{\frac{2r}{B}}\,\sigma^2(x,r)} \tag{6.2}$$

where

$$\xi = x - Br + 2Kr\sigma(x,\theta) - \frac{(M^2+\frac{K}{2})}{B}\int_0^x \sigma(x',\theta)dx' - 2Kr\frac{\partial\tau(x,\theta)}{\partial\theta} - \underline{4\frac{K^2}{B}r\left(\frac{\partial\sigma(x,\theta)}{\partial\theta}\right)^2} + \underline{\frac{K}{B}r\sigma^2(x,\theta)\left\{B^2 - 1 - \frac{K}{2} - [K + \frac{3}{2}M^2 - \frac{2K}{M^2}]\ln 2Br\right\}} \tag{6.3}$$

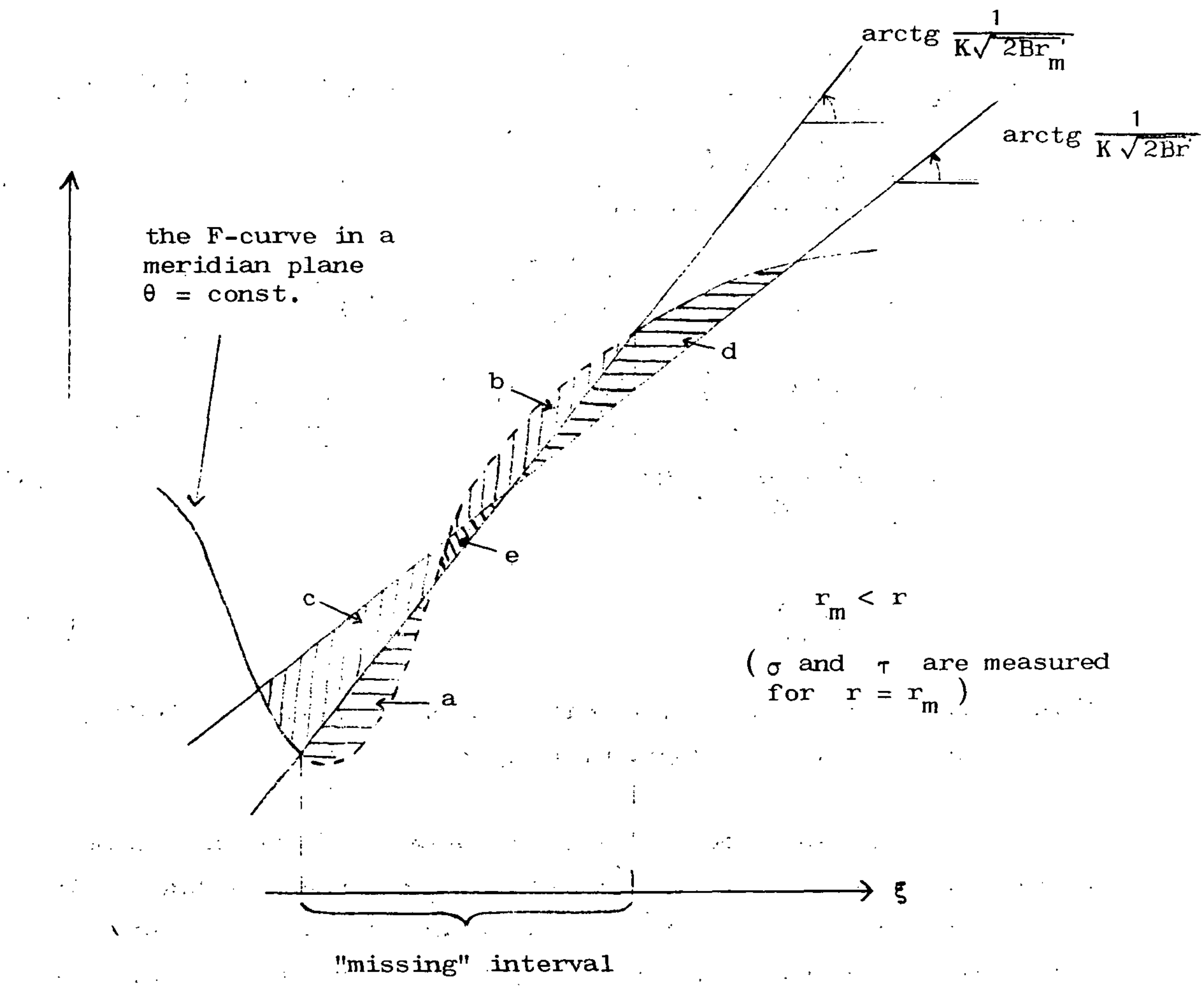
and

$$\theta_o = \theta - \underline{\frac{2K}{B}\frac{\partial\sigma(x,\theta)}{\partial\theta}} \tag{6.4}$$

Here we have omitted the additional terms that arise for a body of revolution with slope discontinuities.

The error in F would be, formally, of order ϵ^{10} if, as assumed in the derivation, $r = \text{ord}(\epsilon^{-4})$. However, when used for moderately large values of Br, the error will of course be larger and primarily governed by the neglected terms in the linear solutions. The terms which are quadratic in σ (underlined in the formulas) are therefore small compared to the others as they essentially represent higher-order cumulative effects for large r, and could therefore usually be ignored. The expressions presented in [14] are then recovered.

If the F-function evaluated in this way is plotted as a function of ξ for constant θ, then the presence of shocks implies that it will be undefined for certain ξ-intervals. However, because the F-function is used only in the study of the far field, one can - as a consequence of Whitham's "rule of equal areas" (see Fig. 6.2) - overcome this difficulty simply by defining the F-curve in each of the "missing" intervals as the straight line connecting the end points of the curve.



$$\left.\begin{array}{l} a = b + e \\ a + c = b + d \end{array}\right\} \Rightarrow \quad c + e = d .$$

Figure 6.2

7. CONCLUSIONS

By the use of characteristic variables a second-order solution valid in the mid field $r = \text{ord}(1)$ of an axisymmetric flow was derived which was found to be identical to the one obtained before [13] by a different method. The mid-field solution was then extended to the near field, $r = O(\varepsilon)$, basically by adding terms equivalent to the triple product term in Van Dyke's [7] solution. Also, correction terms were added to the expressions for the dependent variables so as to make the transformation of the body contour continuous when the body has slope discontinuities. The mid-field solution is valid to first order in the far field, $r = \text{ord}(\varepsilon^4)$; by adding further terms which are of higher order in the near field it was made valid also to second order in the far field. Thus, a second-order solution for axisymmetric flow uniformly valid everywhere has been produced. The solution is expressed directly in terms of the linear solution and may be considered an extension to second order of Whitham's [8] first-order rule. An extension of Whitham's method for constructing the shock wave was also presented and shown to give the shock strength to second order in the mid field. As the new solution also allows the presence of body surface slope discontinuities, it should be of practical value for the calculation of the supersonic non-lifting flow around slender bodies of revolution even in such problems in which the sonic boom is not of primary interest.

For the general three-dimensional case, introduction of characteristic coordinates does not seem to lead to any substantial simplifications for the near and mid fields. For the far field, a second order solution was obtained through the new procedure. The characteristic coördinates used employed the downstream and upstream Mach conoids as surfaces of constant ξ and η, respectively, and perturbed meridian planes ζ = const. intersecting the downstream Mach conoids along bicharacteristics. That the bicharacteristics perhaps are not so fundamental in the three-dimensional case is indicated by the fact that the simplest asymptotic far-field representation, i.e., the one resembling most closely that for axisymmetric flow, is obtained by instead using for the third angular

coordinate θ_o, where $\theta = \theta_o + \Delta\theta$ and where $\Delta\theta$ turns out to be half the angular displacement of the bicharacteristic along the downstream Mach conoid through that point [13].

The asymptotic three-dimensional far-field solution can be put to practical use in at least two obvious ways. First, its simple structure indicates how good approximate solutions can be generated, and some suggestions were given earlier [13] how one could use it for such a purpose. Secondly, it makes possible the generation of simple formulas relating flow deflections in the far field to be Whitham F-function which can be used in a new experimental procedure to determine the sonic boom strength from wind tunnel measurements. Application of this method to a simple body of revolution reported in [14] shows that it is capable of yielding accurate results.

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APPENDIX A

CHARACTERISTICS IN THE AXISYMMETRIC CASE

For axisymmetric flow the system (2.1) - (2.4) becomes

$$\left.\begin{aligned} (U^2 - c^2)\frac{\partial U}{\partial x} + (V^2 - c^2)\frac{\partial V}{\partial r} + UV\left(\frac{\partial V}{\partial x} + \frac{\partial U}{\partial r}\right) &= \frac{c^2 V}{r} \\ \frac{\partial V}{\partial x} = \frac{\partial U}{\partial r} & \end{aligned}\right\} \tag{A.1}$$

When studying such a system, it would certainly be an advantage if x-derivatives and r-derivatives did not appear in the same equation. Therefore, we should be interested in the possibility of finding for example a new set of independent variables ξ, η such that the system could be brought into a form in which one of the equations contained derivatives only with respect to ξ and the other one only with respect to η. Now it is shown in some text-books (see, e.g., [4], p. 101 or [6], p. 433) that in every domain where the flow is supersonic (i.e. $q > c$) the system (A.1) can be brought into such a form, namely

$$\left.\begin{aligned} \frac{\partial U}{\partial \xi}\cot(\vartheta + \alpha) + \frac{\partial V}{\partial \xi} &= \frac{c^2}{V^2 - c^2}\,\frac{V}{r}\,\frac{\partial r}{\partial \xi} \\ \frac{\partial U}{\partial \eta}\cot(\vartheta - \alpha) + \frac{\partial V}{\partial \eta} &= \frac{c^2}{V^2 - c^2}\,\frac{V}{r}\,\frac{\partial r}{\partial \eta} \end{aligned}\right\} \tag{A.2}$$

where the transformation $(\xi,\eta) \rightarrow (x,r)$ is given by

$$\left.\begin{aligned} \frac{\partial x}{\partial \xi} &= \frac{\partial r}{\partial \xi}\cot(\vartheta - \alpha) \\ \frac{\partial x}{\partial \eta} &= \frac{\partial r}{\partial \eta}\cot(\vartheta + \alpha) \end{aligned}\right\} \tag{A.3}$$

and c, ϑ and α are to be found from

$$\left.\begin{aligned} \tfrac{1}{2}(U^2 + V^2) + \frac{c^2}{\gamma - 1} &= \tfrac{1}{2}\hat{q}^2 \\ V &= U \operatorname{tg}\vartheta \\ \frac{1}{M_L} &= \sin\alpha \end{aligned}\right\} \tag{A.4}$$

The system (A.1) is then said to have been put into characteristic form with ξ and η as characteristic coordinates. The curves ξ = const. and η = const. in the xr-plane are known as characteristics (downstream resp. upstream characteristics). (See [4], pp. 62-72, for example).

If we introduce the disturbance velocities u and v and make use of (A.4), then (A.3) becomes (2.5) - (2.6) and (A.2) becomes (2.7) - (2.8).

APPENDIX B

TRANSFORMATION OF THE THREE-DIMENSIONAL FLOW PROBLEM INTO CHARACTERISTIC FORM

With Appendix A in mind it seems reasonable to try to introduce new independent variables ξ, η and ζ such that the system (2.1) - (2.4) is brought into a form in which at least some of the equations would contain derivatives with respect to only two of these variables. The new variables ξ, η and ζ will be called characteristic coordinates.

As the flow is assumed to be irrotational we may introduce a potential function Φ such that $\nabla\Phi = \vec{q}$, or in cylindrical coordinates

$$\left.\begin{aligned} \frac{\partial\Phi}{\partial x} &= U \\ \frac{\partial\Phi}{\partial r} &= V \\ \frac{1}{r}\frac{\partial\Phi}{\partial\theta} &= W \end{aligned}\right\} \tag{B.1}$$

Then combination of (2.1) with (B.1) gives the potential equation

$$\left.\begin{aligned} &(U^2 - c^2)\frac{\partial^2\Phi}{\partial x^2} + (V^2 - c^2)\frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r^2}(W^2 - c^2)\frac{\partial^2\Phi}{\partial\theta^2} + \\ &+ 2VW\frac{1}{r}\frac{\partial^2\Phi}{\partial r\partial\theta} + 2WU\frac{1}{r}\frac{\partial^2\Phi}{\partial\theta\partial x} + 2UV\frac{\partial^2\Phi}{\partial x\partial r} = \frac{V}{r}(c^2 + W^2) \end{aligned}\right\} \tag{B.2}$$

In the following we will use the simplified notations

$$\left.\begin{aligned} x &= x_1 \\ r &= x_2 \\ \theta &= x_3 \end{aligned}\right\} \quad \left.\begin{aligned} U &= q_1 \\ V &= q_2 \\ rW &= q_3 \end{aligned}\right\} \quad \left.\begin{aligned} a_{11} &= U^2 - c^2 \\ a_{22} &= V^2 - c^2 \\ a_{33} &= \frac{1}{r^2}(W^2 - c^2) \end{aligned}\right\} \quad \begin{aligned} a_{12} &= a_{21} = UV \\ a_{13} &= a_{31} = \frac{1}{r}UW \\ a_{23} &= a_{32} = \frac{1}{r}VW \end{aligned}$$

and $h = \frac{V}{r}(c^2 + W^2)$.

We notice that

$$a_{ji} = a_{ij}$$

and

$$\frac{\partial \Phi}{\partial x_i} = q_i .$$

Further, $\bar{\nu}$ and $\frac{\partial}{\partial \bar{\nu}}$ are general notations for the row vectors (ν_1, ν_2, ν_3) and $(\frac{\partial}{\partial \nu_1}, \frac{\partial}{\partial \nu_2}, \frac{\partial}{\partial \nu_3})$, respectively. We also introduce the matrices

$$Q = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}, \qquad H = \begin{pmatrix} h \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad A_1 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} a_{21} & a_{22} & a_{23} \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad A_3 = \begin{pmatrix} a_{31} & a_{32} & a_{33} \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

It is easily seen from the above and Bernoulli's equation that a_{ij}, A_i, h and H are functions of Q and r only.

By means of these notations we can now write the system (2.1) - (2.4) and the equation (B.2) in a more compact form, namely

$$\sum_{i=1}^{3} A_i \frac{\partial Q}{\partial x_i} = H \tag{B.3}$$

and

$$\sum_{i,j=1}^{3} a_{ij} \frac{\partial^2 \Phi}{\partial x_i \partial x_j} = h \tag{B.4}$$

respectively.

With a coordinate transformation

$$y_i = y_i(x_1, x_2, x_3) , \qquad i = 1,2,3.$$

(B.3) and (B.4) — expressed in the new variables y_1, y_2 and y_3 — appear as

$$\sum_{i=1}^{3} B_i \frac{\partial Q}{\partial y_i} = H \tag{B.5}$$

and

$$\sum_{i,j=1}^{3} b_{ij} \frac{\partial^2 \Phi}{\partial y_i \partial y_j} = k \tag{B.6}$$

respectively,

where

$$B_i = \sum_{j=1}^{3} \frac{\partial y_i}{\partial x_j} A_j \tag{B.7}$$

$$b_{ji} = b_{ij} = \sum_{k,l=1}^{3} a_{kl} \frac{\partial y_i}{\partial x_k} \frac{\partial y_j}{\partial x_l} \tag{B.8}$$

and

$$k = h - \sum_{i,j,k,l=1}^{3} a_{ij} \frac{\partial^2 y_k}{\partial x_i \partial x_j} \frac{\partial x_l}{\partial y_k} Q_l \quad .$$

Suppose now that we could find a function $\psi(\bar{x})$ satisfying the first-order equation

$$\sum_{i,j=1}^{3} a_{ij} \frac{\partial \psi}{\partial x_i} \frac{\partial \psi}{\partial x_j} = 0 \tag{B.9}$$

Then by choosing, for example, $y_1 = \psi(\bar{x})$ we would have $b_{11} = 0$

from (B.8) which means absence of $\frac{\partial^2 \Phi}{\partial y_1^2}$ from (B.6). Consequently, if Q were known on only one surface $y_1 = \text{const.}$, it would not be possible to determine $\frac{\partial Q}{\partial y_1}$ uniquely from (B.5) on that surface.

Such a surface is known as a characteristic surface (see [6], Chapt. VI, § 1). Thus the surfaces $\psi(\bar{x}) = \text{const.}$ – where ψ is a solution of (B.9) – form a family of characteristic surfaces.

In our special case we easily find that for arbitrary real numbers p_1, p_2 and p_3

$$\sum_{i,j=1}^{3} a_{ij}\, p_i\, p_j \equiv \Omega_+(\bar{x},\bar{p}) \cdot \Omega_-(\bar{x},\bar{p}) \tag{B.10}$$

where

$$\Omega_\pm(\bar{x},\bar{p}) \equiv Up_1 + Vp_2 + W\frac{p_3}{r} \pm c\sqrt{p_1^2 + p_2^2 + \left(\frac{p_3}{r}\right)^2} \tag{B.11}$$

(Here U, V, W, c and hence also a_{ij} are considered as given functions of $\bar{x}$).

Thus (B.9) can be written

$$\Omega_\pm(\bar{x}, \frac{\partial\psi}{\partial\bar{x}}) = 0 \tag{B.12}$$

From (B.11) we conclude that (B.12) has a solution only if $M_L > 1$. Now we consider only the case with $M_L > 1$ everywhere in the flow field. Evidently (B.9) is a partial differential equation of the first order. Within the general theory of such equations (Monge's theory) (see [4], Ch. II or [5], Ch. 2 or [6], Ch. II) the concept of characteristics plays a very important rôle. The characteristics belonging to (B.9) are called the bicharacteristics of (B.4) (see [6], Ch. VI , § 1). The equation for an arbitrary bicharacteristic follows from Monge's theory and reads

$$\frac{d\bar{x}}{dt} = \frac{\partial\Omega_\pm}{\partial\bar{p}}(\bar{x}, \frac{\partial\psi}{\partial\bar{x}}) \tag{B.13}$$

where t is a curve parameter and ψ satisfies (B.12).

It is easy to verify that ψ is constant along such a curve. Thus every bicharacteristic lies entirely on a characteristic surface ψ = const. One may observe that, in case $\partial\psi/\partial x_1 \neq 0$, equation (B.13) is equivalent to

$$\left.\begin{aligned} \frac{dx_2}{dt} &= \frac{\partial\Omega_\pm}{\partial p_2}\left(\bar{x}, \frac{\partial\psi}{\partial\bar{x}}\right) \\ \frac{dx_3}{dt} &= \frac{\partial\Omega_\pm}{\partial p_3}\left(\bar{x}, \frac{\partial\psi}{\partial\bar{x}}\right) \\ \psi &= \text{const.} \end{aligned}\right\} \qquad (B.14)$$

Consider for a moment the case of undisturbed flow, i.e. $\vec{q} \equiv U_\infty \vec{e}_x$, which is a trivial solution of (2.1) - (2.4).

By putting

$$\left.\begin{aligned} y_1 &= x - B\,r \\ y_2 &= x + B\,r \\ y_3 &= \theta \end{aligned}\right\}$$

one easily finds that

$$\left.\begin{aligned} &\Omega_-\left(\bar{x}, \frac{\partial y_i}{\partial\bar{x}}\right) = 0 , \qquad i = 1,2. \\ \text{and } &\frac{\partial\Omega_-}{\partial p_3}\left(\bar{x}, \frac{\partial y_1}{\partial\bar{x}}\right) = 0 \end{aligned}\right\}$$

Thus (see (B.12) and (B.14) the surfaces y_1 = const. and y_2 = const. are two families of characteristic surfaces, the first of which is intersected along bicharacteristics by the surfaces y_3 = const.

y_1 = const. is known as a <u>downstream Mach cone</u>,

y_2 = const. as an <u>upstream Mach cone</u> and

y_3 = const. is a <u>meridian plane</u>.

Now we return to the general supersonic case, only with the restriction that the disturbance, caused by the body, is considered to be small. In the choice of new independent variables we can then take over some ideas from the special case above.

y_1 is chosen such that:

y_1 = const. is a characteristic surface $\left(\Omega_-(\bar{x}, \frac{\partial y_1}{\partial \bar{x}}) = 0\right)$ and a disturbed downstream Mach cone $(x - Br \approx \text{const.})$.

y_2 is chosen such that:

y_2 = const. is a characteristic surface $\left(\Omega_-(\bar{x}, \frac{\partial y_2}{\partial \bar{x}}) = 0\right)$ and a disturbed upstream Mach cone $(x + Br \approx \text{const.})$.

y_3 is chosen such that:

y_3 = const. intersects y_1 = const. along a bicharacteristic (see (B.15) below) and is a disturbed meridian plane $(\theta \approx \text{const.})$.

From (B.14) we get the condition for y_3 = const. to intersect y_1 = const. along a bicharacteristic, namely

$$dy_3 = dy_1 = 0 \Rightarrow \begin{vmatrix} dx_2 & dx_3 \\ \frac{\partial \Omega_-}{\partial p_2}(\bar{x}, \frac{\partial y_1}{\partial \bar{x}}) & \frac{\partial \Omega_-}{\partial p_3}(\bar{x}, \frac{\partial y_1}{\partial \bar{x}}) \end{vmatrix} = 0$$

or

$$\begin{vmatrix} \frac{\partial x_2}{\partial y_2} & \frac{\partial x_3}{\partial y_2} \\ \frac{\partial \Omega_-}{\partial p_2}(\bar{x}, \frac{\partial y_1}{\partial \bar{x}}) & \frac{\partial \Omega_-}{\partial p_3}(\bar{x}, \frac{\partial y_1}{\partial \bar{x}}) \end{vmatrix} = 0 \qquad \text{(B.15)}$$

Thus, for the determination of the transformation $(x_1, x_2, x_3) \rightarrow (y_1, y_2, y_3)$ we have the following equations:

$$\left\{\begin{aligned}
&U\frac{\partial y_2}{\partial x_1}+V\frac{\partial y_2}{\partial x_2}+\frac{W}{r}\frac{\partial y_2}{\partial x_3}=c\sqrt{\left(\frac{\partial y_2}{\partial x_1}\right)^2+\left(\frac{\partial y_2}{\partial x_2}\right)^2+\frac{1}{r^2}\left(\frac{\partial y_2}{\partial x_3}\right)^2} && \text{(B.16)}\\
&U\frac{\partial y_1}{\partial x_1}+V\frac{\partial y_1}{\partial x_2}+\frac{W}{r}\frac{\partial y_1}{\partial x_3}=c\sqrt{\left(\frac{\partial y_1}{\partial x_1}\right)^2+\left(\frac{\partial y_1}{\partial x_2}\right)^2+\frac{1}{r^2}\left(\frac{\partial y_1}{\partial x_3}\right)^2} && \text{(B.17)}\\
&\frac{\partial x_2}{\partial y_2}\left[U\frac{W}{r}\frac{\partial y_1}{\partial x_1}+V\frac{W}{r}\frac{\partial y_1}{\partial x_2}+\frac{W^2-c^2}{r^2}\frac{\partial y_1}{\partial x_3}\right]=\\
&=\frac{\partial x_3}{\partial y_2}\left[U\,V\frac{\partial y_1}{\partial x_1}+(V^2-c^2)\frac{\partial y_1}{\partial x_2}+V\frac{W}{r}\frac{\partial y_1}{\partial x_3}\right] && \text{(B.18)}
\end{aligned}\right.$$

((B.18) is obtained from (B.15) by means of (B.17)).

We now return to the system (B.5), which is equivalent to the dynamical equations (2.1) - (2.4) expressed in the new variables. Because y_1 = const. and y_2 = const. are characteristic surfaces we get from (B.8) and (B.9) that $b_{11} = b_{22} = 0$. But this means, as mentioned before, that $\partial Q/\partial y_i$ $(i = 1,2)$ cannot be uniquely determined from (B.5) on a surface y_i = const. on which Q is known.

We therefore try to rewrite (B.5) such that one of these equations does not contain $\partial Q/\partial y_1$ and another one does not contain $\partial Q/\partial y_2$. By putting

$$\lambda^{(i)} = \frac{\partial y_i}{\partial \bar{x}}\begin{pmatrix} 1 & -a_{12} & a_{13} & 0\\ 0 & a_{22} & a_{23} & 0\\ 0 & a_{32} & a_{33} & 0\end{pmatrix}, \quad i = 1,2,3,$$

one obtains from (B.7) and (B.9) that

$$\lambda^{(i)}\, B_i = 0 \quad , \qquad i = 1,2.$$

Thus multiplication of (B.5) from the left by $\lambda^{(2)}$ and $\lambda^{(1)}$ gives the scalar equations

$$\left\{\begin{array}{l} \displaystyle\sum_{i=1,3}{}' \left(\lambda^{(2)} B_i\right) \frac{\partial Q}{\partial y_i} = h\,\frac{\partial y_2}{\partial x_1} \qquad \text{(B.19)} \\ \text{and} \\ \displaystyle\sum_{i=2,3}{}' \left(\lambda^{(1)} B_i\right) \frac{\partial Q}{\partial y_i} = h\,\frac{\partial y_1}{\partial x_1} \qquad \text{(B.20)} \\ \text{respectively.} \end{array}\right.$$

To these equations we add the last two from (B.3) expressed in the new variables

$$\left\{\begin{array}{l} \displaystyle\sum_{k=1}^{3} \left(\frac{\partial y_k}{\partial x_3} \frac{\partial}{\partial y_k} \right) q_1 = \sum_{k=1}^{3} \left(\frac{\partial y_k}{\partial x_1} \frac{\partial}{\partial y_k} \right) q_3 \qquad \text{(B.21)} \\ \displaystyle\sum_{k=1}^{3} \left(\frac{\partial y_k}{\partial x_3} \frac{\partial}{\partial y_k} \right) q_2 = \sum_{k=1}^{3} \left(\frac{\partial y_k}{\partial x_2} \frac{\partial}{\partial y_k} \right) q_3 \qquad \text{(B.22)} \end{array}\right.$$

(B.19) - (B.22) are then equivalent to the original system (2.1) - (2.4) expressed in the new variables y_1, y_2 and y_3.

We note that it is not necessary to set up a potential equation (B.2) in order to define the concepts of characteristic surface and bicharacteristic as has been done above. We could just as well work directly with the system (2.1) - (2.4) and obtain the same results, but we follow here the usual approach taken in standard textbooks dealing with second order scalar equations instead of with systems.

Now we put

$$\left.\begin{array}{l} \xi = y_1 \\ \eta = y_2 \\ \zeta = y_3 \end{array}\right\}$$

Noting that

$$\frac{\partial y_i}{\partial x_j} = \frac{\partial(y_1,y_2,y_3)}{\partial(x_1,x_2,x_3)} \begin{vmatrix} \dfrac{\partial x_{j-1}}{\partial y_{i-1}} & \dfrac{\partial x_{j-1}}{\partial y_{i+1}} \\[2ex] \dfrac{\partial x_{j+1}}{\partial y_{i-1}} & \dfrac{\partial x_{j+1}}{\partial y_{i+1}} \end{vmatrix} \qquad i,j \ \text{mod.}3$$

(B.16) — (B.22) can be written

$$\left\{\begin{aligned} &(1+u)\left(\frac{\partial\theta}{\partial\zeta}\frac{\partial r}{\partial\xi}-\frac{\partial\theta}{\partial\xi}\frac{\partial r}{\partial\zeta}\right)+v\left(\frac{\partial x}{\partial\zeta}\frac{\partial\theta}{\partial\xi}-\frac{\partial x}{\partial\xi}\frac{\partial\theta}{\partial\zeta}\right)+\frac{w}{r}\left(\frac{\partial r}{\partial\zeta}\frac{\partial x}{\partial\xi}-\frac{\partial r}{\partial\xi}\frac{\partial x}{\partial\zeta}\right)+\\ &+\frac{1}{M}\left(\frac{c}{c_\infty}\right)\cdot\left\{\left(\frac{\partial\theta}{\partial\zeta}\frac{\partial r}{\partial\xi}-\frac{\partial\theta}{\partial\xi}\frac{\partial r}{\partial\zeta}\right)^2+\left(\frac{\partial x}{\partial\zeta}\frac{\partial\theta}{\partial\xi}-\frac{\partial x}{\partial\xi}\frac{\partial\theta}{\partial\zeta}\right)^2+\right.\\ &\left.+\frac{1}{r^2}\left(\frac{\partial r}{\partial\zeta}\frac{\partial x}{\partial\xi}-\frac{\partial r}{\partial\xi}\frac{\partial x}{\partial\zeta}\right)^2\right\}^{\frac{1}{2}}=0 \end{aligned}\right. \tag{B.23}$$

$$\left\{\begin{aligned} &(1+u)\left(\frac{\partial\theta}{\partial\zeta}\frac{\partial r}{\partial\eta}-\frac{\partial\theta}{\partial\eta}\frac{\partial r}{\partial\zeta}\right)+v\left(\frac{\partial x}{\partial\zeta}\frac{\partial\theta}{\partial\eta}-\frac{\partial x}{\partial\eta}\frac{\partial\theta}{\partial\zeta}\right)+\frac{w}{r}\left(\frac{\partial r}{\partial\zeta}\frac{\partial x}{\partial\eta}-\frac{\partial r}{\partial\eta}\frac{\partial x}{\partial\zeta}\right)-\\ &-\frac{1}{M}\left(\frac{c}{c_\infty}\right)\left\{\left(\frac{\partial\theta}{\partial\zeta}\frac{\partial r}{\partial\eta}-\frac{\partial\theta}{\partial\eta}\frac{\partial r}{\partial\zeta}\right)^2+\left(\frac{\partial x}{\partial\zeta}\frac{\partial\theta}{\partial\eta}-\frac{\partial x}{\partial\eta}\frac{\partial\theta}{\partial\zeta}\right)^2+\right.\\ &\left.+\frac{1}{r^2}\left(\frac{\partial r}{\partial\zeta}\frac{\partial x}{\partial\eta}-\frac{\partial r}{\partial\eta}\frac{\partial x}{\partial\zeta}\right)^2\right\}^{\frac{1}{2}}=0 \end{aligned}\right. \tag{B.24}$$

$$\left\{\begin{aligned} &(1+u)\left(v\frac{\partial\theta}{\partial\eta}-\frac{w}{r}\frac{\partial r}{\partial\eta}\right)\left(\frac{\partial\theta}{\partial\zeta}\frac{\partial r}{\partial\eta}-\frac{\partial\theta}{\partial\eta}\frac{\partial r}{\partial\zeta}\right)+\left(\left[v^2-\frac{1}{M^2}\left(\frac{c}{c_\infty}\right)^2\right]\frac{\partial\theta}{\partial\eta}-\right.\\ &\left.-v\frac{w}{r}\frac{\partial r}{\partial\eta}\right)\left(\frac{\partial x}{\partial\zeta}\frac{\partial\theta}{\partial\eta}-\frac{\partial x}{\partial\eta}\frac{\partial\theta}{\partial\zeta}\right)+\left(v\frac{w}{r}\frac{\partial\theta}{\partial\eta}-\frac{1}{r^2}\left[w^2-\frac{1}{M^2}\left(\frac{c}{c_\infty}\right)^2\right]\frac{\partial r}{\partial\eta}\right)\times\\ &\times\left(\frac{\partial r}{\partial\zeta}\frac{\partial x}{\partial\eta}-\frac{\partial r}{\partial\eta}\frac{\partial x}{\partial\zeta}\right)=0 \end{aligned}\right. \tag{B.25}$$

$$(1+u)\left(v\frac{\partial\theta}{\partial\zeta}-\frac{w}{r}\frac{\partial r}{\partial\zeta}\right)\frac{\partial u}{\partial\xi}+\left(\left[v^2-\frac{1}{M^2}\left(\frac{c}{c_\infty}\right)^2\right]\frac{\partial\theta}{\partial\zeta}-v\frac{w}{r}\frac{\partial r}{\partial\zeta}\right)\frac{\partial v}{\partial\xi}+$$

$$+\left(v\,w\frac{\partial\theta}{\partial\zeta}-\frac{1}{r}\left[w^2-\frac{1}{M^2}\left(\frac{c}{c_\infty}\right)^2\right]\frac{\partial r}{\partial\zeta}\right)\frac{\partial w}{\partial\xi}+$$

$$+(1+u)\left(-v\frac{\partial\theta}{\partial\xi}+\frac{w}{r}\frac{\partial r}{\partial\xi}\right)\frac{\partial u}{\partial\zeta}+\left(-\left[v^2-\frac{1}{M^2}\left(\frac{c}{c_\infty}\right)^2\right]\frac{\partial\theta}{\partial\xi}+v\frac{w}{r}\frac{\partial r}{\partial\xi}\right)\frac{\partial v}{\partial\zeta}+$$

$$+\left(-v\,w\frac{\partial\theta}{\partial\xi}+\frac{1}{r}\left[w^2-\frac{1}{M^2}\left(\frac{c}{c_\infty}\right)^2\right]\frac{\partial r}{\partial\xi}\right)\frac{\partial w}{\partial\zeta}+$$

$$+\frac{v}{r}\frac{1}{M^2}\left(\frac{c}{c_\infty}\right)^2\left(\frac{\partial r}{\partial\zeta}\frac{\partial\theta}{\partial\xi}-\frac{\partial r}{\partial\xi}\frac{\partial\theta}{\partial\zeta}\right)-\left[\frac{\partial(\bar{x})}{\partial(\bar{y})}\sum_{kl}\frac{a_{kl}}{U_\infty^2}\frac{\partial y_1}{\partial x_k}\frac{\partial y_2}{\partial x_l}\right]\cdot\frac{\partial u}{\partial\xi}-$$

$$-\left[\frac{\partial(\bar{x})}{\partial(\bar{y})}\sum_{kl}\frac{a_{kl}}{U_\infty^2}\frac{\partial y_2}{\partial x_k}\frac{\partial y_3}{\partial x_l}\right]\cdot\frac{\partial u}{\partial\zeta}=0 \qquad \text{(B.26)}$$

$$(1+u)\left(v\frac{\partial\theta}{\partial\zeta}-\frac{w}{r}\frac{\partial r}{\partial\zeta}\right)\frac{\partial u}{\partial\eta}+\left(\left[v^2-\frac{1}{M^2}\left(\frac{c}{c_\infty}\right)^2\right]\frac{\partial\theta}{\partial\zeta}-v\frac{w}{r}\frac{\partial r}{\partial\zeta}\right)\frac{\partial v}{\partial\eta}+$$

$$+\left(v\,w\frac{\partial\theta}{\partial\zeta}-\frac{1}{r}\left[w^2-\frac{1}{M^2}\left(\frac{c}{c_\infty}\right)^2\right]\frac{\partial r}{\partial\zeta}\right)\frac{\partial w}{\partial\eta}+$$

$$+(1+u)\left(-v\frac{\partial\theta}{\partial\eta}+\frac{w}{r}\frac{\partial r}{\partial\eta}\right)\frac{\partial u}{\partial\zeta}+\left(-\left[v^2-\frac{1}{M^2}\left(\frac{c}{c_\infty}\right)^2\right]\frac{\partial\theta}{\partial\eta}+v\frac{w}{r}\frac{\partial r}{\partial\eta}\right)\frac{\partial v}{\partial\zeta}+$$

$$+\left(-v\,w\frac{\partial\theta}{\partial\eta}+\frac{1}{r}\left[w^2-\frac{1}{M^2}\left(\frac{c}{c_\infty}\right)^2\right]\frac{\partial r}{\partial\eta}\right)\frac{\partial w}{\partial\zeta}+$$

$$+\frac{v}{r}\frac{1}{M^2}\left(\frac{c}{c_\infty}\right)^2\left(\frac{\partial r}{\partial\zeta}\frac{\partial\theta}{\partial\eta}-\frac{\partial r}{\partial\eta}\frac{\partial\theta}{\partial\zeta}\right)+\left[\frac{\partial(\bar{x})}{\partial(\bar{y})}\sum_{kl}\frac{a_{kl}}{U_\infty^2}\frac{\partial y_1}{\partial x_k}\frac{\partial y_2}{\partial x_l}\right]\frac{\partial u}{\partial\eta}+$$

$$+\left[\frac{\partial(\bar{x})}{\partial(\bar{y})}\sum_{kl}\frac{a_{kl}}{U_\infty^2}\frac{\partial y_3}{\partial x_k}\frac{\partial y_1}{\partial x_l}\right]\cdot\frac{\partial u}{\partial\zeta}=0 \qquad \text{(B.27)}$$

$$\left(\frac{\partial r}{\partial \zeta}\frac{\partial x}{\partial \eta} - \frac{\partial r}{\partial \eta}\frac{\partial x}{\partial \zeta}\right)\frac{\partial u}{\partial \xi} + \left(\frac{\partial r}{\partial \xi}\frac{\partial x}{\partial \zeta} - \frac{\partial r}{\partial \zeta}\frac{\partial x}{\partial \xi}\right)\frac{\partial u}{\partial \eta} +$$

$$+ \left(\frac{\partial r}{\partial \eta}\frac{\partial x}{\partial \xi} - \frac{\partial r}{\partial \xi}\frac{\partial x}{\partial \eta}\right)\frac{\partial u}{\partial \zeta} + r\left(\frac{\partial r}{\partial \zeta}\frac{\partial \theta}{\partial \eta} - \frac{\partial r}{\partial \eta}\frac{\partial \theta}{\partial \zeta}\right)\frac{\partial w}{\partial \xi} +$$

$$+ r\left(\frac{\partial r}{\partial \xi}\frac{\partial \theta}{\partial \zeta} - \frac{\partial r}{\partial \zeta}\frac{\partial \theta}{\partial \xi}\right)\frac{\partial w}{\partial \eta} + r\left(\frac{\partial r}{\partial \eta}\frac{\partial \theta}{\partial \xi} - \frac{\partial r}{\partial \xi}\frac{\partial \theta}{\partial \eta}\right)\frac{\partial w}{\partial \zeta} = 0 \qquad \text{(B.28)}$$

$$\left(\frac{\partial r}{\partial \zeta}\frac{\partial x}{\partial \eta} - \frac{\partial r}{\partial \eta}\frac{\partial x}{\partial \zeta}\right)\frac{\partial v}{\partial \xi} + \left(\frac{\partial r}{\partial \xi}\frac{\partial x}{\partial \zeta} - \frac{\partial r}{\partial \zeta}\frac{\partial x}{\partial \xi}\right)\frac{\partial v}{\partial \eta} +$$

$$+ \left(\frac{\partial r}{\partial \eta}\frac{\partial x}{\partial \xi} - \frac{\partial r}{\partial \xi}\frac{\partial x}{\partial \eta}\right)\frac{\partial v}{\partial \zeta} + r\left(\frac{\partial \theta}{\partial \zeta}\frac{\partial x}{\partial \eta} - \frac{\partial \theta}{\partial \eta}\frac{\partial x}{\partial \zeta}\right)\frac{\partial w}{\partial \xi} +$$

$$+ r\left(\frac{\partial \theta}{\partial \xi}\frac{\partial x}{\partial \zeta} - \frac{\partial \theta}{\partial \zeta}\frac{\partial x}{\partial \xi}\right)\frac{\partial w}{\partial \eta} + r\left(\frac{\partial \theta}{\partial \eta}\frac{\partial x}{\partial \xi} - \frac{\partial \theta}{\partial \xi}\frac{\partial x}{\partial \eta}\right)\frac{\partial w}{\partial \zeta} -$$

$$- w\frac{\partial(x,r,\theta)}{\partial(\xi,\eta,\zeta)} = 0 \qquad \text{(B.29)}$$

In these equations

$$\frac{c}{c_\infty} = \left(1 - \frac{\gamma-1}{2}M^2[2u + u^2 + v^2 + w^2]\right)^{\frac{1}{2}}$$

which is easily obtained from Bernoulli's equation.

APPENDIX C

EXPANSION OF THE LINEARIZED AXISYMMETRIC SOLUTION FOR LARGE r

The solution of (3.25) can be written (see for example [8], p. 307)

$$\begin{cases} u_o(x_o,r_o) = -\displaystyle\int_0^{x_o-Br_o} \frac{f(t)\,dt}{\sqrt{(x_o-t)^2-B^2r_o^{\,2}}} & \text{(C.1)} \\[2em] v_o(x_o,r_o) = \dfrac{1}{r_o}\displaystyle\int_0^{x_o-Br_o} \frac{(x_o-t)\,f(t)\,dt}{\sqrt{(x_o-t)^2-B^2r_o^{\,2}}} & \text{(C.2)} \end{cases}$$

with f given by the integral equation

$$\int_0^{x_o-BR_o(x_o)} \frac{(x_o-t)\,f(t)\,dt}{\sqrt{(x_o-t)^2-B^2R_o^2(x_o)}} = R_o(x_o)\,\omega_o(x_o), \qquad x_o \geq 0 \tag{C.3}$$

where $\omega_o(x_o)$ is of order ε (see 3.47).

Now (C.3) gives for f, approximately,

$$f(t) \approx \frac{d}{dt}\left[R_o(t)\,\omega_o(t)\right]$$

Hence

$$f = O(\varepsilon^2) \tag{C.4}$$

In the following we will use the variables ξ and r_o instead of x_o and r_o.

Then

$$\left.\begin{aligned} u_o &= -\frac{1}{\sqrt{2B\,r_o}} \int_0^\xi \frac{f(t)}{\sqrt{\xi - t}} \left(1 + \frac{\xi - t}{2B\,r_o}\right)^{-\frac{1}{2}} dt \\ B\,u_o + v_o &= \frac{2B}{(2B\,r_o)^{3/2}} \int_0^\xi f(t)\sqrt{\xi - t}\left(1 + \frac{\xi - t}{2B\,r_o}\right)^{-\frac{1}{2}} dt \end{aligned}\right\}$$

For $0 \le \xi < 2B\,r_o$ we can expand $\left(1 + \frac{\xi - t}{2B\,r_o}\right)^{-\frac{1}{2}}$ in a series and integrate term by term.

Thus, observing that

$$\frac{2^n}{(2n-1)!!} \int_0^\xi f(t)\,(\xi - t)^{n-\frac{1}{2}}\,dt = \int_0^\xi dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{n-1}} \left[\int_0^{t_n} \frac{f(t)}{\sqrt{t_n - t}}\,dt\right] dt_n ,$$

we obtain for $0 \le \xi < 2B\,r_o$

$$\left.\begin{aligned} u_o &= -\sum_{n=0}^{+\infty} \frac{(-1)^n}{n!}\left[\frac{(2n-1)!!}{2^n}\right]^2 \frac{F_n^*(\xi)}{(2B\,r_o)^{n+\frac{1}{2}}} \\ v_o &= -B\sum_{n=0}^{+\infty} \frac{2n+1}{2n-1}\,\frac{(-1)^n}{n!}\left[\frac{(2n-1)!!}{2^n}\right]^2 \frac{F_n^*(\xi)}{(2B\,r_o)^{n+\frac{1}{2}}} \end{aligned}\right\} \qquad (C.5)$$

where

$$\left.\begin{aligned} F_n^*(\xi) &= \int_0^\xi F_{n-1}^*(\xi')\,d\xi' \quad (n = 1,2,3,\ldots) \\ F_o(\xi) &= F^*(\xi) = \int_0^\xi \frac{f(t)\,dt}{\sqrt{\xi - t}} \end{aligned}\right\} \qquad (C.6)$$

From (C.4) and (C.6) one easily deduces that

$$F_n^*(\xi) = \underbrace{\left[\frac{2^{n+1}}{(2n+1)!!}\, f(\varkappa_\xi\,\xi)\right]}_{O(\epsilon^2)} \cdot \xi^{n+\frac{1}{2}} \qquad (C.7)$$

for some $\varkappa_\xi$ in the interval $[0,1]$.

APPENDIX D

THE FAR FIELD IN THE AXISYMMETRIC CASE

Assume the expansions

$$\begin{cases} u = \epsilon^4 U_1 + \epsilon^8 U_2 + \dots \\ v = \epsilon^4 V_1 + \epsilon^8 V_2 + \dots \\ x - B\,r \equiv \chi = \chi_o + \epsilon^4 \chi_1 + \dots \\ \epsilon^4 \cdot 2B\,r \equiv H = H_o + \epsilon^4 H_1 + \dots \end{cases}$$

where U_k, V_k, χ_k and H_k are functions of ξ and η, all of order unity and $H_o = \text{ord}\,(1)$.

These are substituted into the system (2.5) - (2.8) which leads to the following set of equations:

To the lowest order we get from (2.5) - (2.8)

$$\begin{cases} \dfrac{\partial H_o}{\partial \xi} = 0 & \text{(D.1)} \\[2ex] \dfrac{\partial \chi_o}{\partial \eta} + \dfrac{1}{2}\dfrac{\partial H_o}{\partial \eta}\left[(M^2 - K)\,U_1 + \dfrac{M^2}{B}\,V_1\right] = 0 & \text{(D.2)} \\[2ex] \dfrac{\partial}{\partial \xi}\,(B\,U_1 + V_1) + \dfrac{V_1}{H_o}\dfrac{\partial H_o}{\partial \xi} = 0 & \text{(D.3)} \\[2ex] \dfrac{\partial}{\partial \eta}\,(B\,U_1 - V_1) - \dfrac{V_1}{H_o}\dfrac{\partial H_o}{\partial \eta} = 0 & \text{(D.4)} \end{cases}$$

Equation (D.1) implies that

$$H_o = h_o(\eta) \tag{D.5}$$

Substitution of (D.1) into (D.3) gives

$$B\,U_1 + V_1 = 0 \tag{D.6}$$

Further, by introducing U_1 from (D.6) into (D.4) we arrive at the equation

$$-2\frac{\partial V_1}{\partial \eta} - \frac{V_1}{H_o}\frac{\partial H_o}{\partial \eta} = 0$$

which has the solution

$$V_1 = B\frac{\mathcal{F}_1(\xi)}{\sqrt{H_o}} \tag{D.7}$$

By means of (D.6) and (D.7) we find from (D.2)

$$\chi_o = \ell_o(\xi) - K\sqrt{H_o}\,\mathcal{F}_1(\xi) \tag{D.8}$$

in which, in order for the solution to take the correct form for zero disturbances, we choose $\ell_o(\xi) = \xi$.

Using (D.5) - (D.8) we obtain from (2.5) - (2.8) to the next order

$$\left\{\begin{aligned}
&\frac{\partial}{\partial \xi}(\chi_o + H_1) = 0 && \text{(D.9)}\\
&\frac{\partial \chi_1}{\partial \eta} = \frac{K}{4B^2}\left[3K - M^2 - \frac{4K}{M^2}\right]\frac{\partial H_o}{\partial \eta}V_1^2 - \frac{K}{2B}\frac{\partial H_1}{\partial \eta}V_1 - \\
&\qquad - \frac{1}{2B}\frac{\partial H_o}{\partial \eta}\left(M^2\,V_2 + [M^2 - K]\,B\,U_2\right) && \text{(D.10)}\\
&\frac{\partial}{\partial \xi}(B\,U_2 + V_2) = -\frac{K}{2B}\frac{\partial V_1^2}{\partial \xi} - \frac{V_1}{H_o}\frac{\partial H_1}{\partial \xi} && \text{(D.11)}\\
&\frac{\partial}{\partial \eta}(B\,U_2 - V_2) = -\frac{1}{H_o}\frac{\partial H_o}{\partial \eta}\frac{M^2 - \frac{K}{2}}{B}V_1^2 + \\
&\qquad + \frac{V_1}{H_o}\frac{\partial H_1}{\partial \eta} - \frac{V_1\,H_1}{H_o^2}\frac{\partial H_o}{\partial \eta} + \frac{V_2}{H_o}\frac{\partial H_o}{\partial \eta} && \text{(D.12)}
\end{aligned}\right.$$

From (D.9) we have

$$H_1 = -\chi_o + h_1(\eta) \tag{D.13}$$

Using (D.7) - (D.9) we get from (D.11)

$$B\,U_2 + V_2 = -\frac{K}{B} V_1{}^2 + \frac{1}{H_o} \int_o^\xi V_1(\xi',\eta)\, d\xi' \tag{D.14}$$

Substitution of U_2 from (D.14) into (D.12) leads by aid of (D.7) to the equation

$$-\frac{2}{\sqrt{H_o}} \frac{\partial}{\partial\eta} (\sqrt{H_o}\, V_2) - \frac{1}{\sqrt{H_o}} \frac{\partial}{\partial\eta} \left(H_1 \frac{V_1}{\sqrt{H_o}}\right) = -\frac{1}{H_o} \frac{\partial H_o}{\partial\eta} \frac{M^2 + \frac{K}{2}}{B} V_1{}^2 +$$

$$+ \frac{3}{2} \frac{1}{H_o{}^2} \frac{\partial H_o}{\partial\eta} \int_o^\xi V_1(\xi',\eta)\, d\xi'$$

from which we find, using (D.7),

$$V_2 = -\frac{M^2 + \frac{K}{2}}{B} V_1{}^2 - \frac{H_1 V_1}{2H_o} + \frac{3}{4} \frac{1}{H_o} \int_o^\xi V_1(\xi',\eta) d\xi' + \frac{B\,\mathcal{F}_2(\xi)}{\sqrt{H_o}} \tag{D.15}$$

By means of (D.7), (D.14) and (D.15) we get from (D.10)

$$\chi_1 = \frac{M^2 - \frac{K}{4}}{B} \int_o^\xi V_1(\xi',\eta) d\xi' + \frac{K}{2B^2} \left(K + \frac{3}{2} M^2 - \frac{2K}{M^2}\right) V_1{}^2\, H_o \ln H_o -$$

$$- \frac{K}{2B} V_1\, H_1 - K \sqrt{H_o}\, \mathcal{F}_2 + \ell_1(\xi) \tag{D.16}$$

Putting

$$\varepsilon^2 \mathcal{F}_1(\xi) + \varepsilon^6 \mathcal{F}_2(\xi) = F(\xi)$$

$$\varepsilon^4\, \ell_1 = \lambda(\xi) + \frac{K}{2} F^2(\xi)\left[M^2 - 2 + \frac{K}{2} + 4\left(K + \frac{3}{2} M^2 - \frac{2K}{M^2}\right) \ln \frac{1}{\varepsilon}\right]$$

and remembering that

$$H_o = H - \varepsilon^4 H_1 + O(\varepsilon^8)$$

we finally obtain from (D.6) – (D.8) and (D.14) – (D.16)

$$u = -\frac{F(\xi)}{\sqrt{2Br}} + \frac{1}{4\cdot(2Br)^{3/2}} \int_0^\xi F(\xi')d\xi' + (M^2 - \frac{K}{2}) \frac{[F(\xi)]^2}{2Br} + O(\varepsilon^{12}) \quad (D.17)$$

$$v = -B u + \frac{B}{(2Br)^{3/2}} \int_0^\xi F(\xi')d\xi' - BK \frac{[F(\xi)]^2}{2Br} + O(\varepsilon^{12}) \quad (D.18)$$

$$x - Br = \xi - K\sqrt{2Br}\, F(\xi) + \frac{M^2 - \frac{K}{4}}{\sqrt{2Br}} \int_0^\xi F(\xi')d\xi' +$$

$$+ \frac{K}{2}(K + \frac{3}{2}M^2 - \frac{2K}{M^2})[F(\xi)]^2 \ln 2Br + \frac{K}{2}(M^2 - 2 + \frac{K}{2})[F(\xi)]^2 +$$

$$+ \lambda(\xi) + O(\varepsilon^8) \quad (D.19)$$

APPENDIX E

DERIVATION OF u_o , v_o , w_o , ϕ_o AND r_o/r AS FUNCTIONS OF σ, τ, AND r IN THE FAR-FIELD DOMAIN $\left(r = \text{ord}\ (\epsilon^{-4})\right)$

By means of (5.31) - (5.33), (5.35), (6.1) and the formulas

$$Bu_o + v_o + \frac{\phi_o}{2r_o} = O(\epsilon^{12}) ,$$

$$\phi = \phi_o - K\, r_o\, u_o\, v_o + O(\epsilon^8) \quad \text{and}$$

$$-Bu + O(\epsilon^8) = -Bu_o + O(\epsilon^8) =$$

$$= v_o + O(\epsilon^8) = v = O(\epsilon^4) ,$$

which can be deduced from (5.17) - (5.22), we obtain

$$v = (1+u)\text{tg}\ \sigma = \left(1 - \frac{v}{B}\right)\sigma + O(\epsilon^{12}) = \left(1 - \frac{1}{B}\left\{\left(1 - \frac{v}{B}\right)\sigma + O(\epsilon^{12})\right\}\right)\sigma + O(\epsilon^{12}).$$

$$\therefore\ v = \left(1 - \frac{1}{B}\sigma\right)\sigma + O(\epsilon^{12}) .$$

$$v_o = \frac{v + O(\epsilon^{12})}{1 + (M^2 + K)\, u_o} = \left(1 + \frac{M^2 + K}{B} v\right) v + O(\epsilon^{12}) .$$

$$\therefore\ v_o = \left(1 + \frac{B^2 + K}{B}\sigma\right)\sigma + O(\epsilon^{12}) .$$

$$\phi = \int_0^x u\, dx' = -\frac{1}{B}\int_0^x v\, dx' + O(\epsilon^8) .$$

$$\therefore\ \phi = -\frac{1}{B}\int_0^x \sigma\, dx' + O(\epsilon^8) .$$

$$\phi_o = \phi + K\, r_o\, u_o\, v_o + O(\epsilon^8) = \phi - \frac{Kr}{B} v^2 + O(\epsilon^8) .$$

$$\therefore\ \phi_o = -\frac{1}{B}\int_0^x \sigma\, dx' - \frac{K}{B} r\, \sigma^2 + O(\epsilon^8) .$$

$$u_o = -\frac{v_o}{B} - \frac{\phi_o}{2Br_o} + O(\epsilon^{12}) = -\frac{v_o}{B} - \frac{\phi_o}{2Br} + O(\epsilon^{12}) ,$$

$$\because \quad u_o = -\frac{1}{B}\left(1 + \frac{B^2 + \frac{K}{2}}{B}\,\sigma\right)\sigma + \frac{1}{2B^2 r}\int_0^x \sigma\, dx' + O(\epsilon^{12}) .$$

$$u = (1 + M^2 u_o)\, u_o + O(\epsilon^{12}) = u_o + \frac{M^2}{B^2} v^2 + O(\epsilon^{12}) .$$

$$\because \quad u = -\frac{1}{B}\left(1 + \frac{-1 + \frac{K}{2}}{B}\,\sigma\right)\sigma + \frac{1}{2B^2 r}\int_0^x \sigma\, dx' + O(\epsilon^{12}) .$$

$$w = (1 + u)\,\mathrm{tg}\,\tau = \tau + O(\epsilon^{12}) .$$

$$\because \quad w = \tau + O(\epsilon^{12}) .$$

$$w_o = w + O(\epsilon^{12}) .$$

$$\because \quad w_o = \tau + O(\epsilon^{12}) .$$

$$\frac{r_o}{r} = \frac{1}{1 - K\, u_o + O(\epsilon^8)} = 1 - \frac{K}{B} v + O(\epsilon^8) .$$

$$\because \quad \frac{r_o}{r} = 1 - \frac{K}{B}\sigma + O(\epsilon^8) .$$

Thus

$$\left.\begin{aligned}
u &= -\frac{1}{B}\left(1 + \frac{-1 + \frac{K}{2}}{B}\,\sigma\right)\sigma + \frac{1}{2B^2 r}\int_0^x \sigma\, dx' + O(\epsilon^{12}) \\
v &= \left(1 - \frac{1}{B}\sigma\right)\sigma + O(\epsilon^{12}) \\
w &= \tau + O(\epsilon^{12}) \\
\phi &= -\frac{1}{B}\int_0^x \sigma\, dx' + O(\epsilon^8)
\end{aligned}\right\} \qquad (E.1)$$

and

$$\left.\begin{aligned}
u_o &= -\frac{1}{B}\left(1 + \frac{B^2 + \frac{K}{2}}{B}\sigma\right)\sigma + \frac{1}{2B^2 r}\int_o^x \sigma\, dx' + O(\epsilon^{12}) \\
v_o &= \left(1 + \frac{B^2 + K}{B}\sigma\right)\sigma + O(\epsilon^{12}) \\
w_o &= \tau + O(\epsilon^{12}) \\
\phi_o &= -\frac{1}{B}\int_o^x \sigma\, dx' - \frac{K}{B} r\,\sigma^2 + O(\epsilon^8) \\
\frac{r_o}{r} &= 1 - \frac{K}{B}\sigma + O(\epsilon^8)
\end{aligned}\right\} \qquad (E.2)$$

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